Check for updates

AUTHOR:
Craig Pournara[1]

AFFILIATION:
[1]Wits School of Education, University of the Witwatersrand, Johannesburg, South Africa

CORRESPONDENCE TO:
Craig Pournara

EMAIL:
craig.pournara@wits.ac.za

DATES:
Received: 04 Apr.2020
Revised: 15 May 2020
Accepted: 18 May 2020
Published: 29 Sep. 2020

HOW TO CITE:
Pournara C. Grade 9 learners' algebra performance: Comparisons across quintiles, insights from errors and curriculum implications. S Afr J Sci. 2020;116(9/10), Art. #8125, 7 pages. https://doi.org/10.17159/sajs.2020/8125

ARTICLE INCLUDES:
☒ Peer review
☐ Supplementary material

DATA AVAILABILITY:
☐ Open data set
☐ All data included
☒ On request from author(s)
☐ Not available
☐ Not applicable

EDITOR:
Jenni Case

KEYWORDS:
learner error, linear equations, didactic cut, integers, quintile 5 maths performance

FUNDING:
South African National Research Foundation (grant numbers 105901, 115261, 71218); First Rand Foundation

Grade 9 learners' algebra performance: Comparisons across quintiles, insights from errors and curriculum implications

It is well known that learners' performance in mathematics in under-resourced secondary schools in South Africa is poor. However, little is known about the performance of learners in top-performing well-resourced secondary schools apart from their good results in Grade 12. In this study, the performance of Grade 9 learners in top-performing quintile 5 (i.e. well-resourced) schools was compared with that of learners in quintile 1–3 (i.e. poorly resourced) schools using a 45-item test. While the quintile 5 learners obtained higher test scores, the performance pattern across the test items was very similar for both quintile groupings. A detailed error analysis of 89 quintile 5 learners' responses to equation items revealed difficulties in applying the standard equation-solving procedure, and in operating with negatives and subtraction, particularly on like terms. These problems may be related to a poorly conceived curriculum in the areas of integers and equations. It is recommended that Grade 8 and 9 teachers pay regular attention to all four arithmetic operations on integers. Furthermore, the teaching of equations should give greater attention to the cognitive shifts required in solving equations with letters on both sides of the equal sign.

Significance:

- Learner performance patterns on a test were similar for learners from top-performing quintile 5 schools and learners from lower quintile schools, although the quintile 5 learners obtained higher test scores.
- Quintile 5 learners' ability to solve linear equations correctly is substantially impacted by their difficulties in simplifying two algebraic terms to a single term, particularly when negatives and/or subtraction are involved.
- Particular aspects of the curriculum may partly be responsible for the difficulties learners experience with integers and equations.

Introduction

Research suggests that Grade 9 learners in quintile 5 schools, which are well resourced, are approximately 4 years ahead of their counterparts in poorly resourced quintile 1–3 schools.[1] This tells us about the relative performance of the two quintile groupings in South African schools and we know only too well about the poor mathematics performance of learners in under-resourced schools[2] but we still do not know much about what is happening at Grade 9 level in well-resourced quintile 5 schools. While many quintile 5 schools produce excellent results at Grade 12 level, research conducted on the mathematics performance of high performers writing the National Benchmark Tests[3] has shown that these learners have difficulties with apparently basic ideas such as percentage and inequalities. Many of these learners will come from quintile 5 and high-fee independent schools. Another indication that all is not well comes from informal discussions with heads of mathematics departments and teachers in quintile 5 schools who despair that many learners in Grades 8 and 9 are not performing at desired levels.

The Wits Maths Connect Secondary (WMCS) project is a research and development project at the University of the Witwatersrand. While our mandate is to focus on teacher professional development in lower quintile schools, we were curious to compare the performance of learners taught by teachers with whom we work, with the performance of Grade 9 learners in quintile 5 schools. We knew the quintile 5 learners would obtain higher marks but we wanted to compare *performance patterns* over the entire test, i.e. the trends in which items had a higher/lower number of correct responses. We also wanted to investigate learner errors and to compare these with previous findings of learners' performance in algebra in lower quintile schools.[4] As the ability to solve equations is fundamental for future success in mathematics, quintile 5 learners' responses to three linear equation items were investigated to gain insight into their fluency in solving equations and also into their fluency in algebraic manipulation. The research was framed by two questions:

- What similarities exist in the test performance patterns of quintile 5 and quintile 1–3 learners?
- What are the most common errors made by quintile 5 learners on linear equation items?

Literature review

Research on learners' approaches to solving linear equations and the errors they make goes back to the late 1980s.[5] There has been a resurgence in research on learner performance on linear equations and notions of equality in recent years.[6-8] One of the key findings of this accumulated research is that learners must be taught formal methods to solve equations of the form $ax+b=cx+d$ because their informal methods, which are adequate for equations of the form $ax+b=c$ break down for equations when there are letters on both sides and/or where there are two terms with letters on one side, e.g. $2x+5–x=4$. This breakdown (or discontinuity) has been referred to as the *didactic cut*[5] and the *cognitive gap*.[9] An equation such as $3x–4=11$, can be solved arithmetically by saying 'what multiplied by 3 and then subtract 4 gives me 11' or $3\times\square–4=11$? Clearly the solution is 5. However, this approach cannot be applied to equations of the form $ax+b=cx+d$ and so learners must be taught to operate on the letters using inverse operations. The initial research on the didactic cut and cognitive gap involved learners who had not yet been taught formal procedures for solving equations. Research conducted with older learners who had already learned equation-solving procedures has challenged the existence of the didactic cut.[7]

Given that the research presented here also involves learners who have been taught procedures for solving equations, I shall rather use the notion of *epistemological obstacle*[10] in speaking about learners' difficulties in making the transition to formal methods for solving equations. An epistemological obstacle involves 'knowledge which functions well in a certain domain of activity and therefore becomes well-established, but then fails to work satisfactorily in another context where it malfunctions'[11]. Thus this notion of obstacle is concerned with a presence rather than an absence of knowledge. With reference to solving equations, the knowledge which has previously worked well refers to arithmetic approaches for solving equations. These methods need to be replaced with new knowledge for solving equations that have letters on both sides (or two terms with letters on one side).

The remainder of the literature review provides an overview of existing research on common errors in solving linear equations. This will provide the reader with the necessary background for the analysis which follows.

Approaches to and errors in solving equations

Kieran[12] identified seven approaches to solving equations, five of which are informal, including *undoing* or *working backwards* and *trial-and-error substitution*. She also distinguished two formal methods: *transposing of terms* (change side, change sign) and *performing the same operation on both sides*. The informal or *arithmetic* methods can be used for equations with letters on one side only while the formal or *algebraic* methods are necessary to solve efficiently equations with letters on both sides.

Four common errors have been identified in solving linear equations. Two of these are the *redistribution error* and *switching addends error*.[12] A redistribution error involves adding a term to one side but subtracting it from the other side. A switching addends error involves 'moving' a term across the equal sign without changing its sign. In this study, I refer to this as a *moving* error and I distinguish between moving constants and moving a letter-term. The *other inverse error*[13] occurs when learners use the incorrect inverse operation, e.g. given $5x=2$, a learner may subtract 5 from both sides instead of dividing by 5, giving $x=-3$ as the solution. Learners making the *familiar structure error*[14] 'force' their answer to fit the form $x=k$ by eliminating additional letters as necessary. For example, a learner who manipulates an equation to obtain $3x=12x$, might first divide by 3 to get $x=4x$ and then drop the letter on the right side and write $x=4$. I refer to this error as *familiar form* because it appears to be driven by learners' desire to produce a final answer of form $x=k$.

Meaning of the equal sign

Learners' conceptions of equality are clearly important in solving equations. Seminal research identified two different views of the equals sign: as a *do something signal* and as an indication of *equivalence.*[15,16] The former operational view is typically associated with unidirectional reasoning about equations and is frequently drawn on to solve equations of form $ax+b=c$. For example, as noted above in the case of $3x-4=11$, the learner reasons 'what multiplied by 3 and then subtract 4 gives me 11?' Here the learner treats the right side as the result of operations performed on the left side. The latter relational view is associated with solving equations of the form $ax+b=cx+d$. Research in the USA found that, across grade levels, learners who demonstrated a relational view of the equal sign, were better able to solve linear equations.[6] However, the authors note that despite learners' inadequate conceptions of equality, attention to the equal sign is typically not addressed in secondary school curricula in the USA. The same is true in the South African secondary curriculum.

Errors in operating on algebraic symbols

A fundamental component of early algebra involves making sense of new symbols and notation. In arithmetic $4+½=4\ ½$ but in algebra one cannot simply juxtapose the two symbols, i.e. $4+a\neq 4a$. In algebra, $a+b$ can be seen as the *process* of adding b to a as well as the resulting *object.*[17] The difficulties in making sense of the new notation explain, to some extent, why learners make errors when working with like and unlike terms, usually conjoining them to produce closure. I work with an expanded notion of conjoining which distinguishes *additive conjoining* from *subtractive conjoining* as two categories of errors that may involve like or unlike terms. While additive conjoining involves addition of positive terms, subtractive conjoining involves negatives and/or subtraction, e.g. $7-x=7x$; $3x-x=3$ and $-x+x=x$.

Errors with negatives

The minus symbol can be viewed as an *operator* (subtraction) or as a *sign* (negative). Hence when learners encounter an equation such as $2+3x=5-2x$, the $-2x$ can be seen as *subtracting* $2x$ or as *negative* $2x$. This duality of the minus symbol poses significant difficulty for learners. Equations involving negatives are more difficult because they are not easily modelled using a balance model.[18] Local research found that learners had greater difficulty in dealing with algebraic expressions when they involve negatives, either as sign or as operation.[4]

Research on subtraction and negatives has identified a range of errors associated with the minus symbol. For example, *right-to-left reasoning* involves subtracting a larger number from a smaller one[18], e.g. $4-7=3$ or $5x-7x=2x$. Overgeneralised integer rules may also lead to errors, e.g. if the multiplication rule is expressed as 'a minus *and* a minus gives a plus', this may lead to the expression $-2x-3x$ being simplified to $+5x$ because explicit attention is not given to the operation. The error of *detaching the minus sign* may occur when learners add numbers or terms with a leading negative[19], e.g. $-2+5=-7$ and $-3x+x=-4x$. In both cases learners detach the minus symbol and isolate it from the expression. They then perform the addition and re-attach the minus symbol to the answer.

Research design and methods

The research reported here stems from a larger study of learner performance. A one-hour test was administered to Grade 9 learners in late September/early October 2018. The first part of the analysis involved a comparison of the performance of these learners with that of pre-existing data from quintile 1–3 schools on the same test. The second part of the study involved further qualitative analysis of the quintile 5 data only with particular focus on learner errors in three test items.

Test instrument

The test consisted of 45 items dealing with number, algebra and function. Most items were typical curriculum items that learners would encounter in text books and tests, and they spanned Grades 7–9 content. The test had been piloted in 2016 with lower quintile schools but we were unsure how the items would perform with a quintile 5 sample. The comparative analysis of both quintile groupings deals with all 45 items. For reference, the broad content area of each question is given in Table 1.

Table 1: Test content areas and question numbers

Content area	Question numbers
Number	1, 2, 3, 4
Algebra	5, 6, 7, 9, 10
Pattern and function	8, 11, 12

Learner responses to each item were coded as *correct*, *incorrect* or *missing*, with no provision for partially correct responses. A learners' test score is simply a count of the number of fully correct responses. The response coding was led by members of the project team, with a group of research assistants. Coding and capturing of responses were moderated – approximately 15% and 20%, respectively. No errors were found in either process.

The error analysis and coding were conducted by the author alone. The focus was on three linear equation items, as shown in Table 2.

Table 2: Linear equation items

Item number	Item
Q9a	$3x$-2=10
Q9b	$3x$-2=4+x
Q9c	2-3x=7-x

Q9a contains a letter on the left side only and can easily be answered using arithmetic approaches. Q9b and Q9c contain letters on both sides but Q9c involves subtraction on both sides of the equation and is therefore more cognitively demanding than Q9b. These items and the associated learners' responses have potential to reveal evidence of learner difficulties in working with equations of the different forms. They also reveal learner errors when the combining of algebraic terms was not the main goal of the manipulations. Consequently, they have potential to reveal errors in solving equations as well as errors in simplifying algebraic expressions. This algebraic work would have been completed in the first half of the year and thus been examined by mid-year.

Sample

The sample from quintile 1–3 schools consisted of 1139 learners from 19 schools, taught by 25 teachers. Schools were selected because their mathematics teachers had completed a professional development course offered by the WMCS project in 2016 or 2017. The selected learners were taught by these teachers in 2018. The quintile 5 sample of 824 learners, taught by 22 teachers, was drawn from four quintile 5 secondary schools, all of which had an existing relationship with the WMCS project. They are top-performing schools in their respective districts and/or top feeder schools to the University of the Witwatersrand. The University's rankings for feeder schools are determined as a ratio of the number of applications to the number of enrolments from that school in a particular year. The large number of teachers is worth noting because it reduces the impact of individual teacher effects on the results.

The sub-sample for the error analysis consisted of 89 learners, across the four quintile 5 schools, who got Q9a correct but Q9b and Q9c incorrect. These criteria suggest possible evidence of an epistemological obstacle in solving more complex linear equations.

Ethical clearance was obtained from the University of the Witwatersrand ethics committee (H17/01/01) and the Gauteng Department of Education (M2017/400AA). All schools were assured that their identity would remain confidential and that no comparisons would be made between schools. Parents and learners were assured that the testing would not impact learners' marks and that they could withdraw at any point. They were also assured of the confidentiality of individual results.

Coding for error analysis

Learner responses were coded according to the approach used and the errors made. Because there are no interview data, it is difficult to infer the underlying reasoning informing learners' written responses. Coding was based on interpretations of what had been written, looking at changes between successive lines of a response together with individualised annotations which learners may have provided such as arrows indicating the moving of a term across the equal sign. While the approaches and errors are reported per item, I also compared each learner's responses to all three items, looking for similarities and differences that might assist in coding their errors.

I distinguished between algebraic and arithmetic approaches to solving the equations. For the purposes of this article, an algebraic approach involves manipulating expressions and operating on the letters. An arithmetic approach involves substitution of a possible solution or an undoing approach. For example, solving $3x$–2=10 by substitution might look as follows: 3(4)–2=10. An undoing approach might be written as: 10+2→12÷3→4

The error analysis was conducted on Q9b and Q9c. Drawing from other analyses of similar data[14,20], I distinguished three broad categories:

1. *Equation errors* – errors in applying inverse operations, collecting like terms and constants on opposites sides of the equal sign, and isolating the letter to determine the solution.
2. *Letter errors* – inappropriate or incorrect execution of operations on terms with letters.
3. *Numeric errors* – operations on constants where the outcome of the operation is incorrect. These are not reported here.

Each category was then sub-divided and errors were allocated specific codes, as described below. I included sub-codes for subtraction/negatives in each of the three categories. I assigned codes to each response based on the three broad categories as well as an 'other' category. It was possible for a single response to have multiple codes. I then dealt systematically with each category identifying sub-codes based both on the literature discussed above and on the data.

Equation error codes

Six sub-codes were identified for equation errors:

1. *Move term with letter* – a term involving a letter is moved unchanged across the equal sign
2. *Move constant* – a constant is moved unchanged across the equal sign
3. *Incorrect inverse* – additive inverse is applied when multiplicative inverse is required, or vice versa
4. *Divide binomial by monomial* – binomial is incorrectly divided by monomial to isolate letter on one side (see Figure 1a)
5. *Familiar format* – inappropriate manipulation of one/both sides of equation to force the form x=k as the final line of the response. This code was only applied when comparing the last two lines of a response, as shown in Figure 1b.

In addition to the five codes above, I included an *incomplete* code for responses where the learner had not produced an answer in the form x=k (see Figure 1c).

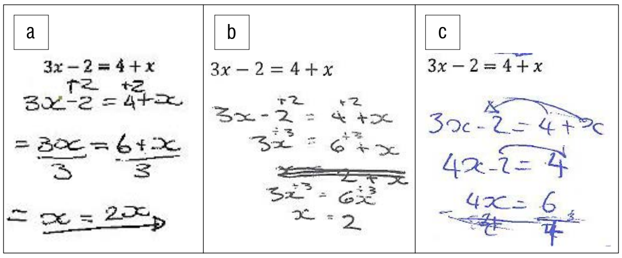

Figure 1: Equation errors: (a) binomial divided by monomial, (b) familiar format and (c) incomplete response.

Letter error codes

Letter errors were distinguished on two dimensions: those involving addition and/or positive terms, and those involving subtraction and/or negatives. The matrix in Table 3 provides examples of typical errors. The examples of subtraction/negatives with like terms include instances of right-to-left reasoning and detaching the minus sign.

Table 3: Examples of letter operation errors

	Like terms	Unlike terms
Addition/positives	$3x+x=3x$	$6+x=6x$ $4+x=5x$
Subtraction/negatives	$3x-x=3x$ $3x-x=3$ $-3x+x=-4x$ $x-3x=2x$	$3x-4=-x$ $7-x=7x$ $3-2x=1$

There were several instances of learners over-generalising exponential laws, e.g. $x+x=x^2$. These errors were separated from the addition-of-like-term errors shown above in order to determine the extent to which learners were still making typical conjoining errors that do not involve exponents. The errors involving exponents were coded as 'other'.

Analysis and results

The analysis is reported in two sections. I begin with the comparison of the overall performance and performance patterns of the quintile 5 group and the quintile 1–3 group. This is followed by the analysis of the responses of the quintile 5 sub-sample to the three equation items.

Overall performance and performance patterns

Table 4 shows that the mean score for the quintile 5 group (24.67) is more than 2.5 times the mean score of the quintile 1–3 group (9.34). This is to be expected and does not merit further discussion. However, a comparison of the performance patterns across the 45 items is of interest (see Figure 2). An obvious difference in the two graphs is that the quintile 5 group performed better than the other group on every item. Again, this is to be expected. More interesting is that the graphs have very similar shapes with peaks and dips in similar places. This suggests that learners across the quintiles find the same questions 'easy' and 'difficult'.

Table 4: Mean scores and standard deviations

Learner group	*N*	Mean	Standard deviation
Quintile 1–3	1139	9.34	6.64
Quintile 5	824	24.67	9.03

Given that the quintile 5 sample is drawn from top-performing schools, it may seem surprising that there were 21 items which fewer than 50% of learners answered correctly. There is a noticeable downward trend in the quintile 5 performance, interspersed with a few peaks. Better performance is generally associated with the numeric items in questions 1 to 4 which focus mainly on integers. Thereafter, most items involve algebra and this is where the downward trend becomes most noticeable. The three highest peaks from question 8 onwards are associated with numeric work: intercepts of the graph of a linear function (Q8a, Q8b); simple linear equation that can be solved without algebraic manipulation (Q9a); and a function machine with numeric inputs and outputs (Q11i, Q11ii). Thus the overall picture of quintile 5 performance is that learners have difficulty with algebraic work and functions. The downward trend is less obvious for the quintile 1–3 group because performance flattens from question 5 onwards, apart from the peaks which occur at similar places to those of the quintile 5 graph.

Three factors must be borne in mind when interpreting the lower than expected performance of the quintile 5 learners. Firstly, learners did not prepare for the test and so the scores merely provide a once-off measure on a particular day. Secondly, the data were collected in late September to early October and, in at least one school, teachers were still completing a section that was included in the test. Consequently, many learners may not yet have consolidated some of the test content. Thirdly, a learner's score indicates completely correct responses and so scores likely will under-estimate learners' knowledge of the test content. For example, a blank response and a response containing a minor slip in a calculation both count as zero.

Given the similarities in the performance pattern of the two groups, there is value in studying the errors made by the quintile 5 group because these will likely provide useful insights for both groups which may in turn lead to recommendations relevant to the teaching of algebra across all quintile schools.

Error analysis of responses to equations items

Table 5 provides a summary of learner performance on the equation items for both groups. In both groups there is a general decrease in performance from Q9a to Q9c with the percentage drop from Q9b to Q9c being larger. The drop in performance between Q9a and Q9b suggests that some learners experience some kind of obstacle in working with equations with letters on both sides. This appears to be exacerbated by the presence of additional negatives in Q9c, particularly given that the letters are being subtracted on both sides of the equation. It is also possible that performance on Q9c was lower because the solution is rational whereas the other two solutions are integers.

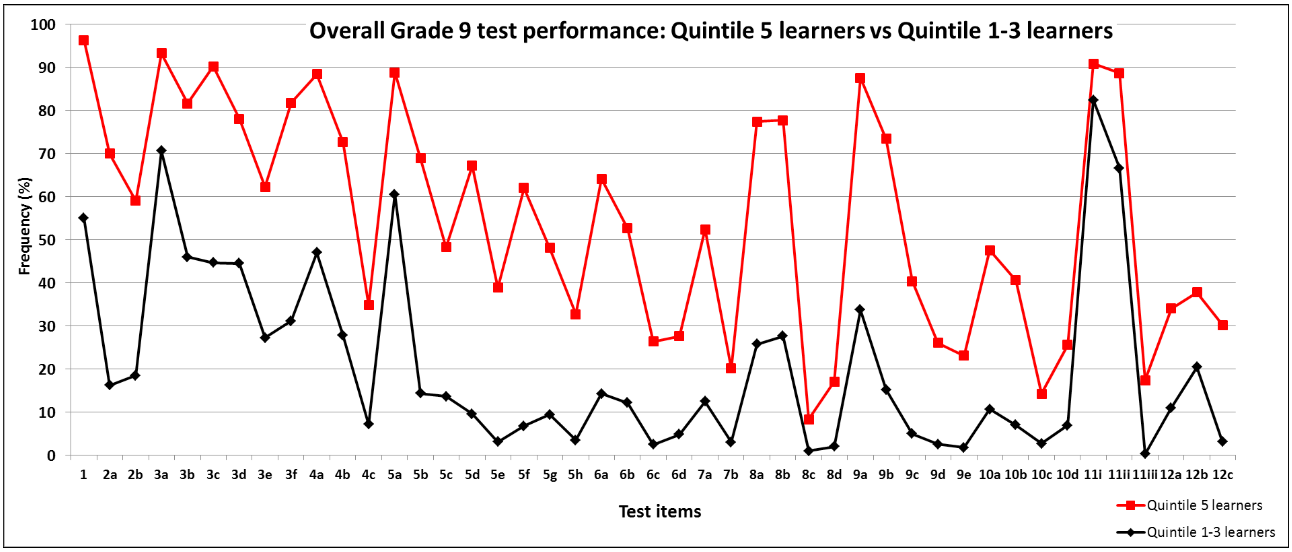

Figure 2: Learners' performance on all test items.

Table 5: Learner performance on linear equation items

Item number	Item	Quintile 5 (N = 824)	Quintile 1–3 (N = 1139)
Q9a	$3x$-2=10	87.6	33.80
Q9b	$3x$-2=4+x	73.4	15.28
Q9c	2-3x=7-x	40.4	5.00

Learners' approaches to solving equations

The analysis from here focuses on the quintile 5 sub-sample. Learners' approaches to Q9a in contrast to their approaches to Q9b and Q9c were of particular interest because this might reveal the extent to which they have overcome the epistemological obstacle described earlier. Table 6 shows that the vast majority used algebraic approaches for Q9a. Of greater interest, however, are the learners who used arithmetic approaches.

Table 6: Approaches to Q9a

Approach	Number of responses	Percentage
Algebraic	63	73.3
Arithmetic Substitution	15	17.4
Undoing	3	3.5
No evidence	5	5.8
Total	**86**	**100**

For those using arithmetic approaches to correctly answer Q9a, it was important to investigate their approaches to Q9b and Q9c: did they shift to an algebraic approach or did they still attempt an arithmetic approach? Eighteen learners successfully used arithmetic approaches for Q9a, either substitution or an undoing approach. Only two of these learners attempted an algebraic approach for Q9b and Q9c.

Of the 15 learners who used a substitution approach for Q9a, 11 used the same approach for Q9b and 9 used it for Q9c. Of the three learners who used an undoing approach for Q9a, two attempted this approach for Q9b and Q9c while the third learner approached both items algebraically. There were five learners who only provided answers and so their responses were coded as 'no evidence'.

In summary, most learners either used all arithmetic approaches or all algebraic approaches. Those adopting arithmetic approaches for all three items constituted 23.8% of the sub-sample. It appears that these learners have not yet recognised the need to reject an inadequate method and replace it with a procedure that makes use of inverse operations. These learners are not operating on the terms with letters in order to solve the equations. It is worth noting that the mean test score for these 15 learners was 14.2 (31.6%) which suggests that they lack algebraic fluency more generally.

I now shift to a discussion of the errors made by the learners who used algebraic approaches for Q9b and Q9c.

Errors in equation operations

A summary of the equation errors is presented in Table 7. A total of 112 equation errors were coded across Q9b and Q9c.

Table 7: Equation errors for Q9b and Q9c

Equation errors	Q9b	Q9c
Move term with letter	14	7
Move constant	6	11
Incorrect inverse	3	2
Divide binomial by monomial	3	3
Force familiar format for solution	13	14
Incomplete	6	13
Other	6	11
Total	**51**	**61**

Not surprisingly, the most common error related to the incorrect use of additive inverses, with 38 errors involving 'moving' a letter/constant across the equal sign without changing sign. It is worth noting that only four learners made 'moving errors' with both numbers and letters. There were five instances where learners used the incorrect inverse, typically subtracting a coefficient instead of dividing. In a small number of instances learners did not collect like terms with letters and so divided a binomial on one side of the equation by the coefficient of the variable on the other side. This typically led to other errors: either the learner performed the division on the numbers only and then conjoined the remaining number and letter (see Figure 1a), or dropped the letter from the expression.

There were a surprising number of responses (27) where learners manipulated the equation to force a familiar format, i.e. $x=k$, as the final line of the response. This took different forms, including dropping/ignoring 'unwanted' letters, e.g. the equation $3x=6x$ became $x=2$.

The high prevalence of *incomplete* responses (19) is closely linked to the familiar format error. In many of these cases, learners did not manipulate their equations to produce a familiar format. Instead they stopped with forms such as $ax=b$, $x=ax$ or $ax/a=b/a$.

The *other* equation errors included converting an equation to an expression, incorrectly combining terms to generate quadratic forms, and various manipulations that did not maintain equality.

Errors in letter operations

A total of 98 letter errors were coded for Q9b and Q9c. As described above, I disaggregated conjoining errors to distinguish (1) additive conjoining (addition/positives) from subtractive conjoining (subtraction/negatives); and (2) operating on like terms from operating on unlike terms. This makes it possible to see more clearly where the majority of errors occurs.

Table 8: Errors in letter operations

	Like terms	Unlike terms	Total
Addition/positives	2	6	**8**
Subtraction/negatives	42	13	**55**
Total	**44**	**19**	**63**

As can be seen in Table 8, only 8 errors involved additive conjoining, irrespective of whether learners were operating on like or unlike terms. By contrast, 55 errors involved subtraction/negatives and 42 of these involved *like* terms. This may be surprising in the light of prior research on conjoining. However, assuming most learners in this sample know we cannot combine unlike terms but can combine like terms, then it is not surprising that the majority of errors occurs when combining like terms. Typical errors included detaching the negative, e.g. $-3x+x=-4x$ and dropping the letter, e.g. $3x-x=3$. There was a surprisingly high number of responses giving $3x-x=3x$. One interpretation is that learners consider only the visible coefficients. In effect they are treating x as $0x$ rather than $1x$. If they separate the numbers and letters, they may reason $3-0=3$ and then append the x to obtain $3x$. However, interviews are necessary to investigate this further.

In addition to the above errors, there were 16 instances in Q9c where learners dropped the negative sign from one line to the next. For example, $-3x$ became $3x$ in the following line. There were also 19 letter errors coded as *other.* These mostly involved over-generalisation of the addition law of exponents which typically led to further errors in attempting to solve an equation that was no longer linear.

Discussion and implications

The overall Grade 9 learner performance on a test of number, algebra and function covering selected Grade 7–9 content was disappointing for both quintile groupings. However, both groupings displayed similar performance patterns. Notwithstanding the caveats mentioned above, a mean score of 54.8% for the quintile 5 group indicates that even towards the end of Grade 9 there are learners in top-performing schools who still have difficulty manipulating algebraic symbols.

Fundamental to the notion of epistemological obstacle in the context of solving equations is that learners accept the need to replace their inappropriate arithmetic approaches with an algebraic approach. Although most learners attempted algebraic approaches for all three items, approximately 24% of the sub-sample used only arithmetic approaches. This may also suggest they do not have a relational view of the equal sign. Furthermore, the error analysis shows that, on average, each learner in the sub-sample made more than one equation error and one letter error across Q9b and Q9c.

The error analysis reveals that errors made by quintile 5 learners in solving linear equations with letters on both sides stem more from difficulties in manipulating algebraic expressions and dealing with negativity than in executing the standard procedure for solving equations. Of the errors reported here, 92 (43.8%) errors relate to negatives/subtraction in some way. Furthermore, nearly half (45.6%) of these negativity errors involved the incorrect simplification of two like terms to a single term. While these findings confirm some of what we found in a previous study on learners' algebra performance in lower quintile schools[4], the insight, at least for quintile 5 schools – that difficulties with negatives and subtraction are more common with *like* terms – is a new empirical finding, although not necessarily surprising.

Implications for curriculum and teaching

From the above findings there are two clear implications for the curriculum and four implications for teaching.

An analysis of the Senior Phase Mathematics curriculum[21] suggests that two problems highlighted in this study may have their roots in the curriculum itself. For example, the content of integers is split over Grades 7 and 8, a split exacerbated by the move from primary to secondary school. The Grade 8 curriculum assumes learners come with knowledge of adding and subtracting integers, that this merely requires revision and that teachers should focus on multiplication and division of integers in Grade 8. While teachers may ignore this 'advice' in their own classrooms, the official teaching support materials such as annual teaching plans, scripted lesson plans and learner workbooks will follow the curriculum closely and thus fall prey to the poorly conceived plans for teaching and learning integers. The evidence from this study and prior research shows clearly that all aspects of integers need detailed attention in Grade 8.

A similar problem arises with equations. There is considerable focus on solving equations by inspection and insufficient attention to formal equation operations in Grade 8. Also, there is no explicit recognition of the importance of attending to equations with letters on both sides. By Grade 9, it is assumed that learners have mastered this work and can move on to more complex linear examples as well as quadratic and exponential equations. Given this breadth of equation types, teachers may overlook the need to deal with simple linear equations with letters on both sides, in the rush to cover the other types. The curriculum needs to foreground the cognitive shifts in moving from equations with a letter on one side to equations with letters on both sides, with additional time allocated to consolidate these procedures, thus supporting learners to navigate and overcome the epistemological obstacle they encounter when they have to operate on the letter in solving equations. The overburdened curriculum could be eased by removing quadratic and exponential equations from Grade 9 as they are dealt with in detail in later years.

Implications for teaching follow closely from the curriculum implications. Firstly, teachers should pay explicit attention to helping learners develop an equivalence view of the equal sign, even in Grades 8 and 9. Without an equivalence view, learners will continue to have difficulty in solving equations of all kinds. Secondly, Grade 8 teachers should pay attention to all four arithmetic operations on integers, with particular attention to subtraction. Attention to fluency with negative numbers should continue into Grade 9. Thirdly, continual attention must be given to fluency in algebraic manipulation, particularly with examples involving subtraction and negatives. This study suggests that such a focus will improve learners' performance on equation solving. Fourthly, teachers need to appreciate the cognitive shift necessary to solve equations with letters on both sides and take time to deal with the case of $ax+b=cx+d$. They should also include equations with more than two terms on each side, e.g. $4-2x+3=3x+1-x$. This provides practice in algebraic simplification as well as in performing inverse operations.

Conclusion

This study shows that difficulties with introductory algebra are not restricted to learners in lower quintile schools. Furthermore, it makes three important contributions. Firstly, there are similarities in the performance patterns of Grade 9 learners across quintiles on a test of number, algebra and function spanning content of Grades 7 to 9. Secondly, it reveals and confirms learners' specific difficulties in working with negatives and subtraction in relation to algebra. Thirdly, it highlights the specific insight that while few learners were making errors with addition of like and unlike terms, there was a proliferation of errors in working with like terms and negatives.

While many learners in quintile 5 schools overcome these difficulties and perform well in mathematics by Grade 12, the same cannot be said for the majority of learners in lower quintile schools. The curriculum recommendations proposed above suggest that specific curriculum changes are necessary in the topics of integers and equations. These may help to address the ways in which the curriculum contributes to learner difficulties with negative numbers and aspects of algebra. The recommendations for teaching address similar issues. However, opportunity for teachers to implement the recommendations requires some flexibility in curriculum pacing to address learners' errors and backlogs.

Acknowledgements

This work is based on research supported by the National Research Foundation (NRF) of South Africa (grant numbers 105901, 115261, 71218) and the First Rand Foundation (FRF). All opinions, findings and conclusions or recommendations are that of the author and the NRF and FRF accept no liability whatsoever in this regard. I am grateful to my colleagues in the WMCS project for many hours of productive discussions on our research and development work, particularly in relation to the teaching and learning of algebra.

Competing interests

I declare that there are no competing interests.

References

1. Spaull N, Kotze J. Starting behind and staying behind in South Africa: The case of insurmountable learning deficits in mathematics. Int J Educ Dev. 2015;41:13–24. https://doi.org/10.1016/j.ijedudev.2015.01.002
2. Reddy V, Van der Berg S, Janse van Rensburg D, Taylor S. Educational outcomes: Pathways and performance in South African high schools. S Afr J Sci. 2012;108(3/4), Art. #620, 8 pages. https://doi.org/10.4102/sajs.v108i3/4.620
3. Bohlmann C, Frith V, Prince R. Mathematical errors made by high performing candidates writing the National Benchmark Tests. Pythagoras. 2017;38(1), Art. #292. https://doi.org/10.4102/pythagoras.v38i1.292
4. Pournara C, Hodgen J, Sanders Y, Adler J. Learners' errors in secondary algebra: Insights from tracking a cohort from Grade 9 to Grade 11 on a diagnostic algebra test. Pythagoras. 2016;37(1), Art. #334. https://doi.org/10.4102/pythagoras.v37i1.334
5. Filloy E, Rojano T. Solving equations: The transition from arithmetic to algebra. For the Learning of Mathematics. 1989;9(2):19–25.
6. Knuth E, Stephens A, McNeil N, Alibali M. Does understanding the equal sign matter? Evidence from solving equations. J Res Math Educ. 2006;37(4):297–312.
7. Lima R, Healy L. The didactic cut in equation solving or a gap between the embodied and the symbolic mathematical worlds? In: Pinto M, editor. Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education. Belo Horizonte, Brazil: PME; 2010. p. 353–360.
8. Roberts A, Le Roux K. A commognitive perspective on Grade 8 and Grade 9 learner thinking about linear equations. Pythagoras. 2018;40(1), Art. #438. https://doi.org/10.4102/pythagoras.v40i1.438
9. Herscovics N, Linchevski L. A cognitive gap between arithmetic and algebra. Educ Stud Math. 1994;27:59–78. https://doi.org/10.1007/bf01284528
10. Brousseau G. Theory of didactical situations in mathematics. Dordrecht: Kluwer; 1997.
11. Cornu B. Limits. In: Tall D, editor. Advanced mathematical thinking. Dordrecht: Kluwer; 1991. p. 153–166.
12. Kieran C. The learning and teaching of school algebra. In: Grouws D, editor. Handbook of research on mathematics teaching and learning. New York: Macmillan; 1992. p. 390–419.
13. Hall R. An analysis of errors made in the solution of simple linear equations. Phil Math Educ J. 2002;15(1):1–67.
14. Sanders Y. Learners' performance in arithmetic equivalences and linear equations Johannesburg: University of the Witwatersrand, 2017.
15. Behr M, Erlwanger S, Nichols E. How children view the equals sign. Math Teach. 1980;92:3–15.
16. Kieran C. Concepts associated with the equality symbol. Educ Stud Math. 1981;12:317–326.
17. Sfard A. On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educ Stud Math. 1991;22:1–36. https://doi.org/10.1007/bf00302715
18. Vlassis J. Making sense of the minus sign or becoming flexible in 'negativity'. Learn Instruct. 2004;14(5):469–484. https://doi.org/10.1016/j.learninstruc.2004.06.012
19. Linchevski L, Herscovics N. Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. Educ Stud Math. 1996;30:39–65. https://doi.org/10.1007/bf00163752
20. Halley A. Linear equations: Grade 9 learners' progress and their difficulties over time. Johannesburg: University of the Witwatersrand; 2019.
21. South African Department of Basic Education (DBE). Curriculum and Assessment Policy Statement Grades 7-9 Mathematics. Pretoria: DBE; 2011.