SUPPLEMENTARY MATERIAL TO: Mashao et al. S Afr J Sci. 2021;117(1/2), Art. #7941.

HOW TO CITE:
Mashao DC, Kosch MJ, Bór J, Nnadih S. The altitude of sprites observed over South Africa [supplementary material]. S Afr J Sci. 2021;117(1/2), Art. #7941. https://doi.org/10.17159/sajs.2021/7941/suppl

Appendix 1: Technique

Estimation of altitude

The cosine and sine rules are defined according to Todhunter[1].

In spherical trigonometry:

Cosine rule

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A)$$

$$\cos(b) = \cos(c)\cos(a) + \sin(c)\sin(a)\cos(B)$$

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$$

Sine rule

$$\frac{\sin(A)}{\cos(a)} = \frac{\sin(B)}{\cos(b)} = \frac{\sin(C)}{\cos(c)},$$

where a, b, c denote sides and A, B, C denote angles of a spherical triangle.

In planar trigonometry:

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\sin(A)$$

$$b^2 = c^2 + a^2 - 2ca\sin(B)$$

$$c^2 = a^2 + b^2 - 2ab\sin(C)$$

Sine rule

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},$$

where a, b, c denote sides and A, B, C denote angles of the triangle.

Figure 2 shows basic triangles on a sphere. We applied spherical trigonometry in the horizontal axes and planar geometry in the vertical axis of Figure 2 in order to derive the equations to calculate *r* and *h*. We define the cosine rule in spherical trigonometry using Figure 2b and solve for A to obtain:

$$\mathrm{A} = \cos^{-1}[\cos(\mathrm{B})\cos(\mathrm{C}) + \sin(\mathrm{B})\sin(\mathrm{C})\,\mathrm{Cos}(\mathrm{AA})] \qquad \text{Equation 1}$$

Then we define the sine rule and cosine rule of side h + R_E and using Figure 2a obtain:

$$(\mathrm{h} + \mathrm{R_E})^2 = \mathrm{R_E^2} + \mathrm{r}^2 + 2\mathrm{r}\mathrm{R_E}\sin\theta \qquad \text{Equation 2}$$

$$\frac{\sin(\mathrm{A})}{\sin(90+\theta)} = \frac{\sin(\mathrm{A})}{\cos(\theta)} = \frac{\mathrm{r}}{\mathrm{h}+\mathrm{R_E}} \qquad \text{Equation 3}$$

By equating (2) and (3) we obtain:

$$\frac{\mathrm{r}^2\cos^2(\theta)}{\sin^2(\mathrm{A})} = \mathrm{R_E}^2 + \mathrm{r}^2 + 2\mathrm{r}\mathrm{R_E}\sin\theta$$

Therefore:

$$\frac{\mathrm{r}^2\cos^2(\theta)}{\sin^2(\mathrm{A})} - \mathrm{R_E}^2 - \mathrm{r}^2 - 2\mathrm{r}\mathrm{R_E}\sin\theta = 0 \qquad \text{Equation 4}$$

We apply the quadratic equation solution to (4) and replace A by the terms on the right side in (1):

$$\mathrm{r} = \frac{2\mathrm{R_E}\sin(\theta) \pm \sqrt{4\mathrm{R_E^2}\sin^2(\theta) + 4\left[\frac{\cos^2(\theta) - \sin^2(\mathrm{A})}{\sin^2(\mathrm{A})}\right]\mathrm{R_E^2}}}{2\left[\frac{\cos^2(\theta) - \sin^2(\mathrm{A})}{\sin^2(\mathrm{A})}\right]} \qquad \text{Equation 5}$$

We then substitute r into (3) to solve for h, and these terms are always positive. We are left with the following two solutions:

$$\mathrm{r} = \frac{2\mathrm{R_E}\sin(\theta) + \sqrt{4\mathrm{R_E^2}\sin^2(\theta) + 4\left[\frac{\cos^2(\theta) - \sin^2(\mathrm{A})}{\sin^2(\mathrm{A})}\right]\mathrm{R_E^2}}}{2\left[\frac{\cos^2(\theta) - \sin^2(\mathrm{A})}{\sin^2(\mathrm{A})}\right]} \qquad \text{Equation 6}$$

$$\mathrm{h} = \left[\frac{2\mathrm{R_E}\sin(\theta) + \sqrt{4\mathrm{R_E^2}\sin^2(\theta) + 4\left[\frac{\cos^2(\theta) - \sin^2(\mathrm{A})}{\sin^2(\mathrm{A})}\right]\mathrm{R_E^2}}}{2\left[\frac{\cos^2(\theta) - \sin^2(\mathrm{A})}{\sin(\mathrm{A})}\right]}\right]\cos(\theta) - \mathrm{R_E} \qquad \text{Equation 7}$$

Finding the direction of sprites procedure

The star fitting was done by employing the horizon coordinate system and equatorial coordinates (right ascension and declination). The horizon coordinate system is required to determine the angular position of any astronomical object in the local sky, in our case a star. The horizon coordinate system is dependent on the time of the observation and the observation site's geographical location.[2,3]

In order to fix the positions of objects in the sky, equatorial coordinates are employed.[2,4] Right ascension and declination of the objects in the celestial sphere are independent of the time of observation and the observer location and remain constant. Right ascension and declination coordinates are equivalent to longitude and latitude coordinates on Earth, respectively. The right ascension is related to a parameter called hour angle. Both right ascension and hour angle are measured in hours, minutes, and seconds.[2,4,5] Hour angle ($\underline{H}$) is expressed mathematically as:

$$\underline{H} = LST - \alpha \qquad \text{Equation 8}$$

where: $\underline{H}$ = hour angle

LST = local sidereal time

α = star right ascension

To calculate $\underline{H}$, one should have the stars' information (right ascension and declination), observer position (latitude, longitude), and the time and date of the observation. α is obtained from the star almanac and LST is obtained from the time of the observation, which is usually measured in universal time (UT). The procedure of converting UT to LST is summarised as follows: First one must convert the observation date into Julian day number and then use the observation time together with the Julian day number to get time in Greenwich sidereal time (GST). Finally, use the observer's longitude and GST to convert time to LST. For more information about time conversion from UT to LST, see Duffett-Smith.[2]

It is possible to calculate stars' azimuth and elevation angles as a function of time for the camera's location. As the time of the sprite event, observer's position, right ascension and declination of stars from the star almanac are known, the stars' azimuth and elevation horizon coordinates are calculated by using Equations 9 and 10:

$$\sin(alt) = \sin\delta \sin\emptyset + \cos\delta \cos\emptyset \cos\underline{H} \qquad \text{Equation 9}$$

$$\cos(AZ) = \frac{\sin\delta - \sin\emptyset \sin(alt)}{\cos\emptyset \cos(alt)} \qquad \text{Equation 10}$$

where: *alt* = star altitude

δ = star declination

$\emptyset$ = observer latitude

AZ = star azimuth

The light rays from distant object stars reach the Earth on a curved path due to the presence of the Earth's atmosphere, which bends the rays, so it is necessary to consider the atmospheric refraction when calculating the position of the stars, especially near the local horizon. However, the refraction angle (*R*) depends on the zenith angle (*z*) which is defined as:

$$z = 90° - alt \qquad \text{Equation 11}$$

If a star elevation is greater than 15°, the appropriate formula for calculating *R* is[2]:

$$R = {0.004552P \ \tan(z)}\Big/_{(273 + T)} \ degrees \qquad \text{Equation 12}$$

where: *P* = barometric pressure in millibars

T = temperature in degrees centigrade

If a star elevation is less than 15°, the appropriate formula for calculating *R* is[2]:

$$R = \frac{P(0.1594+0.0196(alt)+0.00002(alt)^2)}{(273+T)(1+0.505(alt)+0.0845(alt)^2)} \ degrees \qquad \text{Equation 13}$$

The value of *R* is added to alt in order to find the apparent altitude (elevation angle). We used the following two equations to produce the star plot in two dimensions on the image plane in pixel coordinates:

$$y = z \ \cos(AZ) \qquad \text{Equation 14}$$

$$x = z \ \sin(AZ) \qquad \text{Equation 15}$$

References

1. Todhunter I. Spherical trigonometry. 5th ed. Cambridge: Macmillan and Co; 2006.
2. Duffett-Smith P. Practical astronomy with your calculator. 3rd ed. Cambridge: Cambridge University Press; 1989. https://doi.org/10.1017/CBO9780511564895
3. Timeanddate.com. The horizontal coordinate system [webpage on the Internet]. c2019 [cited 2019 Nov 20]. Available from: https://www.timeanddate.com/astronomy/horizontal-coordinate-system.html
4. COSMOS – The SAO encyclopedia of astronomy. Equatorial coordinate system [webpage on the Internet]. c2019 [cited 2019 Nov 20]. Available from: http://astronomy.swin.edu.au/cosmos/E/Equatorial+Coordinate+System
5. COSMOS – The SAO encyclopedia of astronomy. Right ascension [webpage on the Internet]. c2019 [cited 2019 Nov 20]. Available from: http://astronomy.swin.edu.au/cosmos/R/Right+Ascension