Online supplementary material to: Nyabadza F, Winkler D. A simulation age-specific tuberculosis model for the Cape Town metropole. S Afr J Sci. 2013;109(9/10), Art. #2012-0106, 7 pages. http://dx.doi.org/10.1590/sajs.2013/20120106

Appendix 1: State variables and parameters of the compartmental model for TB infection

Class	**Description**
S(a,t)	Susceptibles – those who are at risk of getting the Mtb infection, (i.e. they are not infected or diseased)
V(a,t)	Vaccinated – those who are vaccinated at a very young age
L(a,t)	Latent class – those who are infected but not diseased
I(a,t)	Infectives – those with active disease who are infectious
R(a,t)	Recovered – those who are cured from the active disease either through treatment or self-recovery
Parameter	**Description**
$\mu(a)$	Natural per capita death rate as a function of age
$\beta(a,t)$	Effective contact rate for an individual of age *a* at time *t*
ψ(a)	Rate of vaccination as a function of age
θ(a)	Proportion of individuals who have a reduced risk due to prior vaccination
p(a)	Proportion of individuals who progress slowly to the latent phase
r_1(a)	Re-infection parameter for the latent cases
r_2(a)	Re-infection parameter related to susceptibility after recovery
ρ(a)	Rate of recovery (by treatment or self-recovery)
ϕ(a)	Rate of relapse
δ(a)	Death rate related to TB
σ(a)	Reactivation rate

Appendix 2: The boundary and initial conditions for the population in the Cape Town metropole

The boundary condition for the susceptible at age 0 is

$$S(0,t) = 49300\, exp[0.029(t - 1996)]\,.$$

The recruitment rate is assumed to increase exponentially with time as the Cape Town metropole population grows at about 2.9% per annum.[1] We assume that the recruitment rate increases in accordance with the increase in population over time.

The initial conditions of the classes are also specified.

The susceptible individuals' initial condition at time zero is

$$S(a,0) = \frac{36628}{1 + 0.0185a^2}$$

It is a decreasing function with age because once you are infected you do not return back to the susceptible compartment.

The vaccinated individuals' initial condition is

$$V(a,0) = \frac{146512}{1 + 0.0185a^2}.$$

Likewise, vaccinated individuals' initial condition is a decreasing function with age because once you are infected you do not return back to the vaccinated compartment.

The latently infected individuals' initial condition is

$$L(a,0) = 9508[exp(0.02a) - 1].$$

It is an increasing function of age because once you are infected you do not completely get cured from TB so one can expect that the number of latently infected individuals will only rise as age increases.

The infectious individuals' initial condition is

$$I(a,0) = 1.7482a(100 - a).$$

It is also an increasing function of age but then decreases at about the age of 50 when individuals start to die from aging. As more infectious individuals die from aging there are fewer infectious individuals who remain alive.

The recovered individuals' initial condition is

$$R(a,0) = 2.9752a(100 - a).$$

The number of recovered individuals is an increasing function of age but then decreases at about the age of 50 when individuals start to die from aging. As more recovered individuals die from aging there are fewer recovered individuals who remain alive.

Appendix 3: The finite difference scheme

Numerical scheme

Finite differences are used in the simulations as they are easier to deal with when writing Matlab codes. With these simulations we use a three-point stencil scheme because a stencil with too few points is not accurate while those with too many points cause computational difficulties. If partial differential equations are approximated by finite differences they need to get solved on a rectangular domain with $a \in [0, A]$ and $t \in [0, T]$ where A and T are the maximum age and maximum time under consideration, respectively. In this study, we use backward differences. We divide this rectangular domain into a grid as follows:

$$a_i = ih, \quad (i = 1, \ldots, M),$$
$$t_i = jh, \quad (j = 1, \ldots, N),$$

where h is the grid size so that $h = \frac{A}{M} = \frac{T}{N}$.

We approximate the derivatives as in Shim et al.[2]

$$\begin{aligned}
\frac{\partial X(a,t)}{\partial a} + \frac{\partial X(a,t)}{\partial t} &\approx \frac{\left[X(a_i, t_j) - X(a_{i-1}, t_j)\right] + \left[X(a_i, t_j) - X(a_i, t_{j-1})\right]}{h}, \\
&= \frac{2X(a_i, t_j) - \left[X(a_{i-1}, t_j) + X(a_i, t_{j-1})\right]}{h}, \\
&= \frac{2X_i^j - (X_{i-1}^j + X_i^{j-1})}{h},
\end{aligned}$$

where X is the vector of state variables.

The boundary conditions for the finite difference scheme are

$$S_0^j = \Pi_j \quad \text{and} \quad V_0^j = I_0^j = L_0^j = R_0^j = 0.$$

Because the succeeding value of I_i^j is needed in all the difference equations, we thus use I_i^{j-1} as an approximation to I_i^j in the first calculation of our grids. We then use I_i^j again in the difference equations of the following calculations and run it a number of times until it converges to a solution.

Parameters for each age and year which correspond to the modified data sets were obtained and then connected with each other by means of straight lines to form simple parameter functions. The incidence of active TB disease is given by

$$I_{inc} = \frac{[1 - p(a)]\lambda(a,t)S(a,t) + \sigma(a)L(a,t) + r_1(a)\lambda(a,t)L(a,t) + \phi R(a,t)}{S(a,t) + L(a,t) + R(a,t)}.$$

However, if the force of infection is taken as the integral it will only be a function of time and it will produce a tape-like three-dimensional incidence function. This function is constant with respect to age but varies with respect to time. So, it is therefore desirable to rather make the force of infection a function of both age and time. In discretised form it is written as

$\lambda_i^j := \beta_i^j I_i^j$.

The discretised system of equations appears as follows:

$$\frac{S_i^j - S_{i-1}^j}{h} + \frac{S_i^j - S_i^{j-1}}{h} = -(\lambda_i^j + \mu_i + \psi_i)S_i^j, \qquad \text{Equation 1}$$

$$\frac{V_i^j - V_{i-1}^j}{h} + \frac{V_i^j - V_i^{j-1}}{h} = \psi_i S_i^j - (\theta_i \lambda_i^j + \mu_i)V_i^j, \qquad \text{Equation 2}$$

$$\frac{L_i^j - L_{i-1}^j}{h} + \frac{L_i^j - L_i^{j-1}}{h} = \lambda_i^j\left(p_i S_i^j + \theta_i V_i^j + r_{2_i} R_i^j\right) - \left(r_{1_i}\lambda_i^j + \sigma_i + \mu_i\right)L_i^j, \qquad \text{Equation 3}$$

$$\frac{I_i^j - I_{i-1}^j}{h} + \frac{I_i^j - I_i^{j-1}}{h} = \lambda_i^j(1-p_i)S_i^j + (\lambda_i^j r_{1_i} + \sigma_i)L_i^j + \phi_i R_i^j$$

$$-(\mu_i + \delta_i + \rho_i)I_i^j, \qquad \text{Equation 4}$$

$$\frac{R_i^j - R_{i-1}^j}{h} + \frac{R_i^j - R_i^{j-1}}{h} = \rho_i I_i^j - (r_{2_i}\lambda_i^j + \phi_i + \mu_i)R_i^j. \qquad \text{Equation 5}$$

By solving Equations 1–5, we obtain:

$$S_i^j = \frac{S_{i-1}^j + S_i^{j-1}}{2 + h(\lambda_i^j + \mu_i + \psi_i)}, \qquad \text{Equation 6}$$

$$V_i^j = \frac{h\psi_i S_i^j + V_{i-1}^j + V_i^{j-1}}{2 + h(\theta_i \lambda_i^j + \mu_i)}, \qquad \text{Equation 7}$$

$$L_i^j = \frac{h\lambda_i^j\left(p_i S_i^j + \theta_i V_i^j + r_{2_i} R_i^j\right) + L_{i-1}^j + L_i^{j-1}}{2 + h(r_{1_i}\lambda_i^j + \sigma_i + \mu_i)}, \qquad \text{Equation 8}$$

$$I_i^j = \frac{h[\lambda_i^j(1-p_i)S_i^j + (\lambda_i^j r_{1_i} + \sigma_i)L_i^j + \phi_i R_i^j] + I_{i-1}^j + I_i^{j-1}}{2 + h(\mu_i + \delta_i + \rho_i)}, \qquad \text{Equation 9}$$

$$R_i^j = \frac{h\rho_i I_i^j + R_{i-1}^j + R_i^{j-1}}{2 + h(r_{2_i}\lambda_i^j + \phi_i + \mu_i)}. \qquad \text{Equation 10}$$

In order for us to prove the validity of the difference scheme, we need to establish the convergence of the numerical scheme.[2] We can let the error terms to be

$$\zeta_i^j = S(a_i, t_j) - S_i^j, \qquad \text{Equation 11}$$

$$\nu_i^j = V(a_i, t_j) - V_i^j, \qquad \text{Equation 12}$$

$$\varepsilon_i^j = L(a_i, t_j) - L_i^j, \qquad \text{Equation 13}$$

$$\iota_i^j = I(a_i, t_j) - I_i^j, \qquad \text{Equation 14}$$

$$\chi_i^j = R(a_i, t_j) - R_i^j. \qquad \text{Equation 15}$$

Given that

$$\frac{2S_i^j - S_{i-1}^j - S_i^{j-1}}{h} = \frac{2S_i^j - 2S(a_i, t_j) - S_{i-1}^j + S(a_{i-1}, t_j) - S_i^{j-1} + S(a_i, t_{j-1})}{h}$$

$$+ \frac{2S(a_i, t_j) - S(a_{i-1}, t_j) - S(a_i, t_{j-1})}{h}, \qquad \text{Equation 16}$$

we have

$$\frac{2S_i^j - S_{i-1}^j - S_i^{j-1}}{h} = -\frac{2\zeta_i^j - \zeta_{i-1}^j - \zeta_i^{j-1}}{h} + \frac{2S(a_i,t_j) - S(a_{i-1},t_j) - S(a_i,t_{j-1})}{h}.$$

So

$$\frac{2\zeta_i^j - \zeta_{i-1}^j - \zeta_i^{j-1}}{h} = -\frac{2S_i^j - S_{i-1}^j - S_i^{j-1}}{h} + \frac{2S(a_i,t_j) - S(a_{i-1},t_j) - S(a_i,t_{j-1})}{h}. \quad \text{Equation 17}$$

Substituting the expression in Equation 1 in Equation 17 gives

$$\frac{2\zeta_i^j - \zeta_{i-1}^j - \zeta_i^{j-1}}{h} = (\lambda_i^j + \mu_i + \psi_i)S_i^j + \text{Q}$$

$$= (\lambda_i^j + \mu_i + \psi_i)[S_i^j - S(a_i,t_j)] + (\lambda_i^j + \mu_i + \psi_i)S(a_i,t_j) + Q$$

$$= -(\lambda_i^j + \mu_i + \psi_i)\zeta_i^j + (\lambda_i^j + \mu_i + \psi_i)S(a_i,t_j) + Q,$$

where $Q = \dfrac{2S(a_i,t_j) - S(a_{i-1},t_j) - S(a_i,t_{j-1})}{h}$.

If we say $O(h) = \lim_{h \to 0^+}\left[(\lambda_i^j + \mu_i + \psi_i)S(a_i,t_j) + \dfrac{2S(a_i,t_j) - S(a_{i-1},t_j) - S(a_i,t_{j-1})}{h}\right],$

then we have

$$\frac{2\zeta_i^j - \zeta_{i-1}^j - \zeta_i^{j-1}}{h} = -(\lambda_i^j + \mu_i + \psi_i)\zeta_i^j + O(h) \quad \text{Equation 18}$$

Similarly, for the other equations, we obtain:

$$\frac{2\nu_i^j - \nu_{i-1}^j - \nu_i^{j-1}}{h} = \psi_i\zeta_i^j - (\theta\lambda_i^j + \mu_i)\nu_i^j + O(h), \quad \text{Equation 19}$$

$$\frac{2\varepsilon_i^j - \varepsilon_{i-1}^j - \varepsilon_i^{j-1}}{h} = \lambda_i^j(p\zeta_i^j + \theta\nu_i^j + r_2\chi_i^j) - (r_1\lambda_i^j + \sigma + \mu_i)\varepsilon_i^j + O(h), \quad \text{Equation 20}$$

$$\frac{2\iota_i^j - \iota_{i-1}^j - \iota_i^{j-1}}{h} = \lambda_i^j(1-p)\zeta_i^j + (\lambda_i^j r_1 + \sigma)\varepsilon_i^j + \phi\chi_i^j - (\mu_i + \delta + \rho)\iota_i^j + O(h), \quad \text{Equation 21}$$

$$\frac{2\chi_i^j - \chi_{i-1}^j - \chi_i^{j-1}}{h} = \rho\iota_i^j - (r_2\lambda_i^j + \phi + \mu_i)\chi_i^j + O(h). \quad \text{Equation 22}$$

By taking initial and boundary conditions of the system to be

$\zeta_0^j = \nu_0^j = \varepsilon_0^j = \iota_0^j = \chi_0^j = 0$ for $j = 1,2,\ldots,N$ and $\zeta_i^0 = \nu_i^0 = \varepsilon_i^0 = \iota_i^0 = \chi_i^0 = 0$ for $i = 1,2,\ldots,M$ respectively, we can solve Equations 19–22 to obtain solutions for $\zeta_i^j, \nu_i^j, \varepsilon_i^j, \iota_i^j$ and χ_i^j.

$$\zeta_i^j = \frac{\zeta_{i-1}^j + \zeta_i^{j-1}}{2 + h(\lambda_i^j + \mu_i + \psi_i)} + O(h^2), \qquad \text{Equation 23}$$

$$\nu_i^j = \frac{h\psi_i\zeta_i^j + \nu_{i-1}^j + \nu_i^{j-1}}{2 + h(\theta\lambda_i^j + \mu_i)} + O(h^2), \qquad \text{Equation 24}$$

$$\varepsilon_i^j = \frac{\lambda_i^j h(p\zeta_i^j + \theta\nu_i^j + r_2\chi_i^j) + \varepsilon_{i-1}^j + \varepsilon_i^{j-1}}{2 + h(r_1\lambda_i^j + \sigma + \mu_i)} + O(h^2), \qquad \text{Equation 25}$$

$$\iota_i^j = \frac{h[\lambda_i^j(1-p)\zeta_i^j + (\lambda_i^j r_1 + \sigma)\varepsilon_i^j + \phi\chi_i^j] + \iota_{i-1}^j + \iota_i^{j-1}}{2 + h(\mu_i + \delta + \rho)} + O(h^2), \qquad \text{Equation 26}$$

$$\chi_i^j = \frac{h\rho\iota_i^j + \chi_{i-1}^j + \chi_i^{j-1}}{2 + h(r_2\lambda_i^j + \phi + \mu_i)} + O(h^2). \qquad \text{Equation 27}$$

We shall make use of a norm on the error estimates which has the form

$$\| f_n \| = \sum_{i=1}^{i=N} f_i h.$$

From Equations 23–27 we get:

$$\| \zeta^j \| \le \frac{1}{2}(\| \zeta^j \| + \| \zeta^{j-1} \|) + O(h^2),$$

$$\| \nu^j \| \le h\psi_i \| \zeta^j \| + \frac{1}{2}(\| \nu^j \| + \| \nu^{j-1} \|) + O(h^2),$$

$$\| \varepsilon_i^j \| \le \beta_i^j \| \iota^j \| h(p\| \zeta^j \| + \theta \| \nu^j \| + r_2 \| \chi^j \|) + \frac{1}{2}(\| \varepsilon^j \| + \| \varepsilon^{j-1} \|) + O(h^2),$$

$$\| \iota^j \| \le h[\beta_i^j \| \iota^j \| (1-p) \| \zeta^j \| + (\beta_i^j \| \iota^j \| r_1 + \sigma) \| \varepsilon^j \| + \phi \| \chi^j \|] + \frac{1}{2}(\| \iota^j \| + \| \iota^{j-1} \|) + O(h^2),$$

$$\| \chi^j \| \le h\rho \| \iota^j \| + \frac{1}{2}(\| \chi^j \| + \| \chi^{j-1} \|) + O(h^2).$$

As $h \to 0$ the error estimates approach zero. Thus Equations 23–27 converge to zero as $h \to 0$. Thus the proof of convergence validates the finite difference scheme.

References

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