Online supplementary material to: Chikobvu D, Chifurira R. Modelling of extreme minimum rainfall using generalised extreme value distribution for Zimbabwe. S Afr J Sci. 2015;111(9/10), Art. #2014-0271, 8 pages. http://dx.doi.org/10.17159/sajs.2015/20140271

Appendix 1: Maximum likelihood estimation of the generalised extreme value distribution of annual rainfall data for Zimbabwe from 1901 to 2009

The probability density function of the generalised extreme value distribution (GEVD) is:

$$g_{\xi,\mu,\sigma}(x) = \frac{1}{\sigma}\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1-\frac{1}{\xi}} exp\left\{-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}}\right\}$$

The log-likelihood of the GEVD when $\xi \neq 0$ is:

$$l(\mu,\sigma,\xi) = -mlog\sigma - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}$$

Differentiating l with respect to ξ:

To differentiate $\sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}$ we let

$$y = \sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}$$

taking natural logarithm on both sides:

$$Iny = In\sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}$$

$$= \sum_{i=1}^{m} In\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}$$

$$= -\frac{1}{\xi}\sum_{i=1}^{m} In\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]$$

Now we differentiate implicitly

$$\frac{1}{y}\frac{dy}{d\xi} = \frac{1}{\xi^2}\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \frac{1}{\xi}\sum_{i=1}^{m}\frac{x_i-\mu}{\sigma}.\frac{1}{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)}$$

$$= \frac{1}{\xi^2}\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \frac{1}{\xi}\sum_{i=1}^{m}\frac{x_i-\mu}{\sigma+\xi(x_i-\mu)}$$

therefore:

$$\frac{dy}{d\xi} = y\left[\frac{1}{\xi^2}\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \frac{1}{\xi}\sum_{i=1}^{m}\frac{x_i-\mu}{\sigma+\xi(x_i-\mu)}\right]$$

$$= \sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\left[\frac{1}{\xi^2}\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \frac{1}{\xi}\sum_{i=1}^{m}\frac{x_i-\mu}{\sigma+\xi(x_i-\mu)}\right]$$

Now:

$$\frac{dl}{d\xi} = \frac{1}{\xi^2}\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{x_i-\mu}{\sigma}.\frac{1}{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)}$$
$$- \sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\left[\frac{1}{\xi^2}\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]\right.$$
$$\left.- \frac{1}{\xi}\sum_{i=1}^{m}\frac{x_i-\mu}{\sigma}.\frac{1}{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)}\right]$$

$$\frac{dl}{d\xi} = \frac{1}{\xi^2}\sum_{i=1}^{m} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{x_i-\mu}{\sigma+\xi(x_i-\mu)} - \frac{1}{\xi^2}\sum_{i=1}^{m}\sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} log\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right] + \frac{1}{\xi}\sum_{i=1}^{m}\sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\left[\frac{x_i-\mu}{\sigma+\xi(x_i-\mu)}\right] \quad \text{Equation 1}$$

Differentiating l with respect to σ:

$$\frac{dl}{d\sigma} = -\frac{m}{\sigma} - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{1}{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)}.(-\xi)\left(\frac{x_i-\mu}{\sigma^2}\right)$$
$$-\sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\left[\left(-\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{1}{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)}.(-\xi)\left(\frac{x_i-\mu}{\sigma^2}\right)\right]$$

$$\frac{dl}{d\sigma} = -\frac{m}{\sigma} - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{\xi(x_i-\mu)}{\sigma[\sigma+\xi(x_i-\mu)]} - \sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}.\sum_{i=1}^{m}\frac{\xi(x_i-\mu)}{\sigma[\sigma+\xi(x_i-\mu)]}$$

$$\frac{dl}{d\sigma} = -\frac{m}{\sigma} - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{\xi(x_i-\mu)}{\sigma[\sigma+\xi(x_i-\mu)]} - \sum_{i=1}^{m}\sum_{i=1}^{m}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}.\frac{\xi(x_i-\mu)}{\sigma[\sigma+\xi(x_i-\mu)]}$$ Equation 2

Differentiating l with respect to μ:

$$\frac{dl}{d\mu} = -\left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{1}{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)}.\left(-\frac{\xi}{\sigma}\right) - \left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\left[\left(-\frac{1}{\xi}\right)\frac{1}{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)}.\left(-\frac{\xi}{\sigma}\right)\right]$$

$$\frac{dl}{d\mu} = \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{\xi}{\sigma+\xi(x_i-\mu)} - \left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\left[\frac{1}{\sigma+\xi(x_i-\mu)}\right]$$ Equation 3

To obtain maximum likelihood estimates of ξ, σ and μ, Equations 1, 2 and 3 are set to zero and standard numerical optimisation algorithms are applied.

Calculating the return levels

Generalised extreme value distribution

Let $p = \frac{1}{T}$ where T is the return period in years

$$G_{\xi,\mu,\sigma}(x) = exp\left\{-\left(1+\xi\left(\frac{x_p-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}}\right\} = 1-p$$

$$\ln(1-p) = -\left(1+\xi\left(\frac{x_p-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}}$$

$$-\ln(1-p) = \left(1+\xi\left(\frac{x_p-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}}$$

$$(-\ln(1-p))^{-\xi} - 1 = \xi\left(\frac{x_p-\mu}{\sigma}\right)$$

$$\frac{\sigma}{\xi}\left((-\ln(1-p))^{-\xi} - 1\right) + \mu = x_p$$

Normal distribution

$$g_{\mu,\sigma} = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{1}{2\sigma^2}(x_p-\mu)^2\right\}$$

$$G_{\mu,\sigma} = \int_{-\infty}^{x_p} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{1}{2\sigma^2}(x_p-\mu)^2\right\} dx = 1-p$$

$$\phi\left(\frac{x_p-\mu}{\sigma}\right) = G_{\mu,\sigma}$$

$$\phi\left(\frac{x_p-\mu}{\sigma}\right) = 1-p$$

$$\phi^{-1}\phi\left(\frac{x_p-\mu}{\sigma}\right) = \phi^{-1}(1-p)$$

$$\left(\frac{x_p-\mu}{\sigma}\right) = \phi^{-1}(1-p)$$

$$x_p - \mu = \sigma\phi^{-1}(1-p)$$

$$x_p = \sigma\phi^{-1}(1-p) + \mu$$

$$x_p = \sigma Z_{1-p} + \mu$$

where $0 < p < 1$ and $p = \frac{1}{T}$

Appendix 2: Maximum annual rainfall data

Year	Maximum rainfall (mm)		Year	Maximum rainfall (mm)		Year	Maximum rainfall (mm)		Year	Maximum rainfall (mm)
1901	879.1		1928	851.2		1955	762.5		1983	464
1902	492.7		1929	628.9		1956	691.7		1984	745
1903	692.7		1930	567.7		1957	795.7		1985	695.4
1904	634.7		1931	724.9		1958	656.4		1986	422.4
1905	616.7		1932	591.2		1959	483.4		1987	744
1906	805.9		1933	565.3		1960	727.9		1988	605.3
1907	665.8		1934	668.4		1961	612.4		1989	625.1
1908	780.2		1935	584.6		1962	787.9		1990	501.6
1909	646.8		1936	655.7		1963	467.1		1991	335.3
1910	802.9		1937	552.9		1964	509.2		1992	629.7
1911	433.2		1938	976.7		1965	570.5		1993	519.3
1912	549.7		1939	703.8		1966	677.4		1994	418.8
1913	473.2		1940	626.9		1967	404.8		1995	700.5
1914	953.5		1941	500.8		1968	716		1996	801.6
1915	394.3		1942	787.6		1969	538.7		1997	532
1916	566.7		1943	731.3		1970	577.2		1998	778.9
1917	1118.1		1944	601.1		1971	806		1999	883.5
1918	726.6		1945	734.7		1972	371.1		2000	728.6
1919	743.2		1946	365.2		1973	1003.5		2001	465.8
1920	701.3		1947	765.3		1974	819.9		2002	609.5
1921	385		1948	535		1975	736.7		2003	712.3
1922	936.5		1949	518.9		1976	748.4		2004	529
1923	399		1950	516.8		1978	569.3		2005	835.7
1924	1192.6		1951	784.3		1979	640.2		2006	598.4
1925	753.1		1952	908.6		1980	860.7		2007	796.2
1926	512.6		1953	581.6		1981	439.7		2008	734.9
1927	553.6		1954	1012.5		1982	403.1		2009	709.1