Peer review history for:

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The original manuscript for review is appended below.

Reviewer 1: Round 1
Date completed: 17 March 2025
Conflicts of interest: None

Recommendation: Accept / Revisions required / Resubmit for review / Resubmit elsewhere / Decline / See comments
REVIEWER: Revisions required
Does the review fall within the scope of SAJS? Yes/No
REVIEWER: Yes
Is the review written in a style suitable for a non-specialist and is it of wider than only specialist interest? Yes/No
REVIEWER: Yes
Do the Title and Abstract clearly and accurately reflect the content of the manuscript? Yes/No
REVIEWER: Yes
Does the review provide a significantly novel perspective or significant recent advances in the field? Yes/No
REVIEWER: Yes
Is the objective of the review concisely stated? Yes/No
REVIEWER: Yes
Is appropriate and adequate reference made to other work in the field? Yes/No
REVIEWER: Yes
Do current debates and points of contention receive appropriate coverage? Yes/No/Not applicable
REVIEWER: Yes
Are gaps in the literature adequately identified? Yes/No/Not applicable
REVIEWER: Yes
Does the review provide direction for future research? Yes/No/Not applicable
REVIEWER: Yes

Are the methodology and statistical treatment appropriate? Not applicable/Yes/No/Partly/Not qualified to judge
REVIEWER: Yes
Are the interpretations and recommendations aligned with the objective? Yes/Partly/No
REVIEWER: Yes
Please rate the manuscript on overall contribution to the field Excellent/Good/Average/Below average/Poor
REVIEWER: Good
Please rate the manuscript on language, grammar and tone Excellent/Good/Average/Below average/Poor
REVIEWER: Good
Is the manuscript concise and free of repetition and redundancies? Yes/No
REVIEWER: Yes
Is the supplementary material relevant and separated appropriately from the main document? Yes/No/Not applicable
REVIEWER: Not applicable
Please rate the manuscript on overall quality Excellent/Good/Average/Below average/Poor
REVIEWER: Good
If accepted, would you recommend that the article receives priority publication? Yes/No
REVIEWER: Yes
Are you willing to review a revision of this manuscript? Yes/No
REVIEWER: Yes
With regard to our policy on 'Publishing peer review reports', do you give us permission to publish your anonymised peer review report alongside the authors' response, as a supplementary file to the published article? Publication is voluntary and only with permission from both yourself and the author. Yes/No
REVIEWER: Yes
Comments to the Author: The author(s) did a mathematical model for bioconvection flow in porous media with microbial exoprotein. They came up with a system of seven nonlinear partial differential equations (pdes) that govern the model. Some appropriate boundary conditions were prescribed. Similarity transformations were used to transform the system of defining pdes into a dimensionless system of nonlinear pdes which were then solved numerically used the spectral quasi-linearization method. The reviewer is quite pleased with how the author(s) presented their work. All the sections were thoroughly and diligently done. The manuscript can be accepted for publication in sajs subject to a few suggested minor revisions. 1. Authors to clearly state the novelty of the current work. 2. Revise line 79 to read: u and v are the velocity components along the x and y axis, respectively. 3. Present all the parameters and their units if any, in a table of nomenclature. 4. Define the terms $\boldsymbol{R_f}$, $\boldsymbol{R\_\theta}$, etc. 5. Revise line 242. It seems to be incorrect.

Author response to Reviewer 1: Round 1

Authors to clearly state the novelty of the current work.
AUTHOR: The novelty has been clearly outlined in the manuscript.
Revise line 79 to read: u and v are the velocity components along the x and y axis, respectively.
AUTHOR: The correction has been made in the text.
Present all the parameters and their units if any, in a table of nomenclature.
AUTHOR: Nomenclature has been included in the manuscript.
Define the terms R_f, R_θ, etc.
AUTHOR: The terms have been defined in the manuscript.
Revise line 242. It seems to be incorrect.
AUTHOR: The correction has been made in the manuscript.

Reviewer 2: Round 1

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Reviewer 1: Round 2
Date completed: 22 July 2025
Conflicts of interest: None

Recommendation: Accept / Revisions required / Resubmit for review / Resubmit elsewhere / Decline / See comments
REVIEWER: Accept
Does the review fall within the scope of SAJS? Yes/No
REVIEWER: Yes
Is the review written in a style suitable for a non-specialist and is it of wider than only specialist interest? Yes/No
REVIEWER: Yes
Do the Title and Abstract clearly and accurately reflect the content of the manuscript? Yes/No
REVIEWER: Yes
Does the review provide a significantly novel perspective or significant recent advances in the field? Yes/No
REVIEWER: Yes
Is the objective of the review concisely stated? Yes/No
REVIEWER: Yes
Is appropriate and adequate reference made to other work in the field? Yes/No
REVIEWER: Yes
Do current debates and points of contention receive appropriate coverage? Yes/No/Not applicable
REVIEWER: Yes
Are gaps in the literature adequately identified? Yes/No/Not applicable
REVIEWER: Yes
Does the review provide direction for future research? Yes/No/Not applicable
REVIEWER: Yes

Are the methodology and statistical treatment appropriate? Not applicable/Yes/No/Partly/Not qualified to judge
REVIEWER: Yes
Are the interpretations and recommendations aligned with the objective? Yes/Partly/No
REVIEWER: Yes
Please rate the manuscript on overall contribution to the field Excellent/Good/Average/Below average/Poor
REVIEWER: Good
Please rate the manuscript on language, grammar and tone Excellent/Good/Average/Below average/Poor
REVIEWER: Good
Is the manuscript concise and free of repetition and redundancies? Yes/No
REVIEWER: Yes
Is the supplementary material relevant and separated appropriately from the main document? Yes/No/Not applicable
REVIEWER: Not applicable
Please rate the manuscript on overall quality Excellent/Good/Average/Below average/Poor
REVIEWER: Good
If accepted, would you recommend that the article receives priority publication? Yes/No
REVIEWER: Yes
Are you willing to review a revision of this manuscript? Yes/No
REVIEWER: Yes
With regard to our policy on 'Publishing peer review reports', do you give us permission to publish your anonymised peer review report alongside the authors' response, as a supplementary file to the published article? Publication is voluntary and only with permission from both yourself and the author. Yes/No
REVIEWER: Yes
Comments to the Author: The authors addressed all the issues and corrections suggested by the reviewers and they were thorough. The manuscript is now perfect and I recommend it for publication in the *South African Journal of Science*.

Reviewer 2: Round 2
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Reviewer 2: Round 3
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A mathematical model for bioconvection flow in porous media with microbial exoprotein activity

Abstract

Microbes play a pivotal role in heat and mass transfer in boundary layer flow. These microbes are added to the nanofluids to promote efficiency and the desired fluid properties. Some microbes excrete exoproteins into the solution that significantly affect the chemical reaction, heat and mass transfer. Based on an extensive literature review, we found no models that take into account the existence of exoprotein dynamics when modelling boundary layer flow. In our model, we developed a system of partial differential equations and included exoprotein dynamics. The system of partial differential equations was transformed into a system of ordinary differential equations and then solved numerically using the spectral quasilinearization method to a high degree of accuracy. Our results show that thermophoresis and the Brownian motion parameters increased the exoprotein concentration. The increase in the growth rate, the carrying capacity, and the exoprotein production rate causes a significant increase in the exoprotein, and eventually, the product increases. The degradation of the exoprotein decreases the product concentration. The increase in the growth rate and the thermophoresis parameters also caused the Nusselt number to increase as well.

Keywords: exoprotein, bioconvection, microbes, spectral quasilinearization method

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Introduction

The study of bioconvection flow is of paramount significance in industrial processes and biomedical technology. Nanofluids and microbes play a significant role in improving flow dynamics about momentum, heat, and mass transfer, enhancing the efficiency of the flow. Bioconvection is the directional collective swimming of microorganisms resulting in a macroscopic convection movement of the fluid[1,2]. Nanofluid is a term coined by Choi[3] to describe fluids that contain nano-sized particles to enhance certain key properties of the fluid for a specific desired function. Although Choi received much recognition for the idea. Maxwell[4] was the first to attempt the technology. This failed because it clogged fluid passages and damaged the pumps, a major setback in the commercialisation of the technology[3,5,6]. Typical nanoparticles usually consist of stable metals such as *Al, Cu, Au, Ag, and Fe*; oxides, namely *Al2O3, CuO, TiO2, and SiO2*; carbides *SiC*; nitrides *AlN and SiN*; or non-metals such as graphite and carbon nanotubes[7,8]. Typical base fluids are water, ethylene glycol (including other coolants), oil (including other lubricants), polymers, bio-fluids, and conductive fluids[7].

Motile microbes in porous media play a pivotal role in the bioconvection flow as they enhance the thermal efficiency of many systems[9]. The microbes can be any of the following: bacteria, fungi, algae,

viruses, protozoa, etc. The microbes swim to the upper surface of the fluid and form a macroscopic, dense layer of microbes in the nanofluid, which becomes unstable[10]. Microbes are considered relatively less dense than water, resulting in bioconvection phenomena. They present an uneven and unstable structure due to self-propelled microbes swimming to the upper layer of the liquid, and this process of bioconvection flow occurs mainly with bacteria[11]. Microbe's classification is based on their response to the type of stimulus, for example, gyrotactic, oxytactic, gravitactic, or phototactic[12]. These microbes produce and actively secrete exoproteins[13]. Another microbial toxin class is endotoxins, which remain within the bacterial outer membrane. The idea of using exoproteins commercially is encountered in the brewing[14, 15] and pharmaceutical industries[16-18]. In this study, we want to analyse the effects of the exoproteins on bioconvection flow. We aim to develop a model that describes and predicts the effects of exoproteins on bioconvection flow. This study is significant considering the impact of exoproteins in cancer treatment when bacteria is exploited to kill advanced cancer cells or alter the function of the cells[14, 15]. Exoproteins are also important in the manufacturing of vaccines and cheese production[16, 17]. Ahmad *et al*[13] developed a model from the literature that described exoprotein production, but no studies have modelled the bioconvection flow that included exoprotein production.

Mathematical model

We consider a two-dimensional bioconvectional flow of nanofluid flowing through a porous medium. The model includes the exoproteins whose concentration at the boundary is m_w and is proportional to the microbe's concentration n_w at the surface, that is $m_w \propto n_w$. The exoprotein produces a solute (a chemical product of interest) whose concentration is given by S and S_w is the concentration of the solute at the wall. T_w and C_w are considered the temperature and nanofluid concentration at the boundary, respectively.

$$\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}u - \frac{\nu}{K}u - \frac{C_b}{x\sqrt{K}}u^2 + (1 - C_\infty)g\beta_T(T - T_\infty) -$$

$$\frac{\rho_c - \rho}{\rho}g\,\beta_c(C - C_\infty) - \frac{\rho_n - \rho}{\rho}g\,\beta_n(n - n_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(1 + \frac{16\sigma^* T_\infty^3}{3\kappa}\right)\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma B_0^2}{\rho C_p}u^2 + \frac{\nu}{KC_p}u^2 + \frac{C_b}{x\sqrt{K}}u^3 +$$

$$\tau\left(D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + D_n\frac{\partial n}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right), \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}, \tag{4}$$

$$u\frac{\partial n}{\partial x} + v\frac{\partial n}{\partial y} + \frac{bW_c}{\Delta C}\frac{\partial}{\partial y}\left(n\frac{\partial C}{\partial y}\right) = D_n\frac{\partial^2 n}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} + r(n - n_\infty)\left(1 - \frac{n - n_\infty}{K_n\,\Delta n}\right), \quad (5)$$

$$u\frac{\partial m}{\partial x} + v\frac{\partial m}{\partial y} = D_m\frac{\partial^2 m}{\partial y^2} + \lambda(n - n_\infty) - \mu\,(m - m_\infty), \quad (6)$$

$$u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = D_S\frac{\partial^2 S}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} + \lambda_s(m - m_\infty), \quad (7)$$

The associated boundary conditions are given as

$$At\ y = 0,\ \ u = cx, -\kappa_f\frac{\partial T}{\partial y} = h_f(T_w - T), D_B\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\frac{\partial T}{\partial y} = 0, m = m_w, S = S_w,$$

$$-\kappa_n\frac{\partial n}{\partial y} = h_n(n_w - n).$$

$$At\ y \to \infty, u \to 0, \frac{\partial u}{\partial y} \to 0, T \to T_\infty, C \to C_\infty, n \to n_\infty, m \to m_\infty, S \to S_\infty. \quad (8)$$

u is the velocity component along the *x*-axis, and v is along the *y*-axis, with y considered to be normal to the surface[9]. ρ is the fluid density, ρ_c is the nanoparticle density, ρ_n is the microbe density, ν is the kinetic viscosity, σ is the fluid conductivity, σ^* is the Stephan- Boltzman constant, which refers to the electrical conductivity of the nanofluid. B_0 is the magnetic field, K is the permeability parameter, g is the gravitational acceleration, β_T is the thermal coefficient of expansion, β_C is the solute coefficient of expansion, β_n is the microbe coefficient of expansion, c_p is the specific heat capacity, C_b is the geometry-dependent drag coefficient, C and C_∞ are the nanoparticle concentration and ambient nanoparticle concentration, respectively, and n and n_∞ are the microbe concentration and ambient microbe, respectively. T and T_∞ is the temperature and ambient temperature, m and m_∞ exoprotein and ambient exoprotein, D_B is the solute Brownian diffusion coefficient, D_T is the thermophoretic diffusion coefficient, D_m is the Brownian diffusion coefficient of the exoprotein, D_n is the Brownian diffusion coefficient of the microbe, κ is the thermal conductivity, $\propto$ is the thermal diffusivity such that $\propto = \kappa/(\rho c_p)$, τ is the ratio of heat capacitance of nanoparticles to the base fluid, b is the chemotaxis constant, W_c is the maximum cell swimming speed, λ_n is the production rate of exoproteins, $\mu_m m$ is the natural decay due to temperature, p is the pressure of fluid. $r(n - n_\infty)(1 - (n - n_\infty)/\Delta n K_n)$ describes the rate of microbe population change, where *r* is the maximum microbial growth rate and K_n is the population carrying capacity. h_f is the convective heat transfer coefficient. h_n is the microbe's transfer rate, and κ_f is the conductivity of the fluid.

Transformation of PDEs

Dimensionless transformations are used to reduce the system of partial differential equations 2 - 7 and the boundary conditions 8 to obtain a system of ordinary differential equations. The transformations are given by:

$$\psi = \sqrt{c\nu}xf(\eta), \eta = y\sqrt{\frac{c}{\nu}}, \theta(\eta) = \frac{T - T_\infty}{T_W - T_\infty}, \phi(\eta) = \frac{C - C_\infty}{C_W - C_\infty}, \chi(\eta) = \frac{n - n_\infty}{n_W - n_\infty}, \xi(\eta) = \frac{m - m_\infty}{m_W - m_\infty},$$

$$\varphi(\eta) = \frac{S - S_\infty}{S_W - S_\infty}, u = cxf', v = -\sqrt{c\nu}f. \quad (9)$$

The transformed equations are given by

$$f''' + ff'' - Mf' - K_p f' - F_r f'^2 + \frac{G_r}{Re^2}[\theta - N_r\phi - R_b\chi] = 0, \quad (10)$$

$$\left(1 + \frac{4}{3}R_d\right)\theta'' + P_r\left[f\theta' + E_c f''^2 + ME_c f'^2 + K_p E_c f'^2 + F_r E_c f'^3 + N_b\theta'\phi' + N_n\chi'\theta' + N_t\theta'^2\right] = 0, \quad (11)$$

$$\phi'' + S_c f\phi' + \frac{N_t}{N_b}\theta'' = 0, \quad (12)$$

$$\chi'' + S_b f\chi' - P_b[\phi'\chi' + \tau_\circ\phi'' + \phi''\chi] + \frac{N_t}{N_n}\theta'' + \tilde{r}S_b\chi\left(1 - \frac{\chi}{K_n}\right) = 0, \quad (13)$$

$$\xi'' + S_d\left[f\xi' - A_2\xi + A_{1\chi}\right] = 0, \quad (14)$$

$$\varphi'' + S_s\left[f\varphi' + \tilde{\lambda}_s\xi\right] + \frac{N_t}{N_s}\theta'' = 0. \quad (15)$$

and the boundary conditions are given as

$$f'(0) = 1, f(0) = 0, f'(\infty) \to 0, f''(\infty) \to 0, \theta'(0) = -B_i\left(1 - \theta(0)\right), \theta(\infty) \to 0, N_b\phi'(0) + N_t\theta'(0) = 0,$$

$$\phi(\infty) \to 0, \chi'(0) = -B_n\left(1 - \chi(0)\right), \chi(\infty) \to 0, \xi(0) = 1, \xi(\infty) \to 0, \varphi(0) \to 1, \varphi(\infty) \to 0. \quad (16)$$

where

$$F_r = \frac{C_b}{\sqrt{K}}, K_p = \frac{\nu}{cK}, Re = \frac{cx^2}{\nu}, G_r = \frac{g\beta_T(1 - C_\infty)(T_w - T_\infty)}{\nu^2}x^3, M = \frac{\sigma\beta_\circ^2}{\rho c}, N_r = \frac{\beta_c(\rho_c - \rho)\Delta C}{\beta_T\rho(1 - C_\infty)\Delta T}, P_r = \frac{\nu}{\alpha},$$

$$R_b = \frac{\beta_n(\rho_n - \rho)\Delta n}{\beta_T\rho(1 - C_\infty)\Delta T}, E_c = \frac{(cx)^2}{c_p\Delta T}, S_c = \frac{\nu}{D_B}, S_b = \frac{\nu}{D_n}, R_d = \frac{4\sigma^* T_\infty^3}{\kappa}, N_b = \frac{\tau D_B\Delta C}{\nu}, N_n = \frac{\tau D_n\Delta n}{\nu}, \tau_\circ = \frac{n_\infty}{\Delta n},$$

$$N_s = \frac{\tau D_s\Delta S}{\nu}, N_t = \frac{\tau D_T\Delta T}{\nu T_\infty}, A_1 = \frac{\lambda\Delta n}{c\Delta m}, A_2 = \frac{\mu_m}{c}, P_b = \frac{bW_c}{D_n}, S_d = \frac{\nu}{D_m}, B_i = \frac{h_f}{\kappa_f}\frac{x}{\sqrt{Re}}, B_n = \frac{h_n}{\kappa_n}\frac{x}{\sqrt{Re}},$$

$$S_s = \frac{\nu}{D_s}, \tilde{\lambda}_s = \frac{\lambda_s\Delta m}{c\Delta S}, \tilde{r} = \frac{r}{c}. \quad (17)$$

We define the parameters in equations 10 - 16 as follows: M is the magnetic field parameter, Re is the Reynolds number (ratio of inertial forces to viscous forces), G_r is the Grashof number (ratio of

buoyancy to viscous force), K_p is the permeability parameter, F_r is called the Forchheimer number, N_r is the buoyancy ratio parameter (ratio of thermal buoyancy to solute buoyancy), N_b is the Brownian motion parameter, which measures the random movement of particles in a fluid. N_n is the bioconvection Brownian motion parameter, which measures the random movement of microbes in a fluid, N_t is the thermophoresis parameter, which is a measure of the movement of microscopic particles due to a force of the temperature gradient. R_b is the bioconvection Rayleigh number (the ratio of thermal transport due to microorganisms to thermal transport via convection), and E_c is the Eckert number, which measures the relationship between a flow's kinetic energy and the boundary layer enthalpy difference. S_c is the Schmidt number (ratio of momentum diffusivity to mass diffusivity), S_b is the biconvection Schmidt number (ratio of momentum diffusivity to microbes diffusivity), S_d is the Schmidt number (ratio of momentum diffusivity to exoprotein diffusivity), S_s is the Schmidt number for the solute(product), R_d is the thermal radiation (relative contribution of conduction heat transfer to thermal radiation transfer), P_r is the Prandtl number (ratio of momentum diffusivity to thermal diffusivity), P_b is the biconvection Peclet number (ratio of the movement of microbes as a result of swimming to movement of microbes due to diffusion), τ_o is the constant microbial concentration difference parameter, τ_m is the constant exoprotein concentration difference parameter, A_1 is the exoprotein production rate by bacteria, A_2 is the natural degradation of exoprotein, λ_s is the production rate for the product "solute", B_i is the Biot number(thermal), and B_n is the Biot number(microbial).

Momentum, Heat, and Mass Coefficients

In this section, we consider the quantities that have an impact at the boundary, such as the local skin friction coefficient, the local Nusselt number, the Sherwood number, and the local density number of microbes. The local skin friction coefficient (C_f) is given by

$$C_f = \frac{\tau_w}{\frac{1}{2}U_w^2} = -2\,Re^{-\frac{1}{2}} f''(0), \tag{18}$$

the local Nusselt number is given as

$$Nu_x = \frac{x\,q_w}{\kappa(T_w - T_\infty)} = -\left(1 + \frac{4}{3}Rd\right) Re^{\frac{1}{2}}\,\theta'(0), \tag{19}$$

the local Sherwood number is given as

$$Sh_x = \frac{x\,q_c}{D_B\,(C_w - C_\infty)} = -Re^{\frac{1}{2}}\phi'(0), \tag{20}$$

the local microbe density number is given as

$$Nn_x = \frac{x\,q_n}{D_n\,(n_w - n_\infty)} = -Re^{\frac{1}{2}}\chi'(0), \tag{21}$$

the local exoprotein density number is given as

$$Nm_x = \frac{x\, q_m}{D_m\,(m_w - m_\infty)} = -Re^{\frac{1}{2}}\xi'(0), \tag{22}$$

and the local product density number is given by

$$Ns_x = \frac{x\, q_s}{D_s\,(S_w - S_\infty)} = -Re^{\frac{1}{2}}\varphi'(0), \tag{23}$$

where

$$\tau_w = -\mu\frac{\partial u}{\partial y}\,|_{y=0}, q_w = \alpha\left(1 + \frac{4\,\sigma^* T_\infty^3}{3\kappa^*\rho C_p}\right)\frac{\partial T}{\partial y}\Big|_{y=0}, q_c = -D_B\frac{\partial C}{\partial y}\Big|_{y=0}, q_n = -D_n\frac{\partial n}{\partial y}\Big|_{y=0},$$

$$q_m = -D_m\frac{\partial m}{\partial y}\Big|_{y=0}, q_s = -D_s\frac{\partial S}{\partial y}\Big|_{y=0} \tag{24}$$

Numerical simulation

The spectral quasilinearization method involves transforming the domain $\eta \in [0, L_x]$ into the computational domain $z_i \in [-1,\ 1]$. The $N+1$ collocation points or grid points are generated from the Gauss-Lobatto [23]

$$z_i = cos\left(\frac{\pi i}{N}\right),\ \ i = 0, 1, 2, \cdots, N \tag{25}$$

so that the grid points are given by

$$\eta_i = \frac{L_x(1+z_i)}{2},\quad i = 0, 1, 2, \cdots, N \tag{26}$$

By the Spectral collocation method, we introduce the differentiation matrix D used to approximate derivatives of unknown variables (such as $f(\eta)$) at the collocation points such that

$$f' = \frac{df}{d\eta} = \sum_{j=0}^{N}\boldsymbol{D}_{ij}f(\eta_j) = Df\,,\quad i = 0, 1, 2, \cdots, N \tag{27}$$

where $\boldsymbol{D} = \frac{2D}{L_x}$ and

$$f = [f(z_o),\ f(z_1),\ \cdots,\ f(z_N)] \tag{28}$$

is the vector function at the collocation points [23]. This implies that

$$f^k = \boldsymbol{D}^k f \tag{29}$$

where k is the order of the derivative.

The solution is approximated using the Lagrange interpolation polynomials of the form

$$u(\eta) = \sum_{t=0}^{N} \hat{u}(\eta_i) L_i(\eta) \quad (30)$$

where

$$L_i(\eta) = \prod_{i=0, i\neq k}^{N} \frac{\eta - \eta_i}{\eta_k - \eta_i}$$

We also define equations 10 - 15 using the functions F, Θ, Φ, ψ, X, and Υ respectively such that:

$$F = f''' + ff'' - Mf' - K_p f' - F_r f'^2 + \frac{G_r}{Re^2}[\theta - N_r\phi - R_b\chi], \quad (31)$$

$$\Theta = \left(1 + \frac{4}{3}R_d\right)\theta'' + P_r\left[f\theta' + E_c f''^2 + ME_c f'^2 + K_p E_c f'^2 + F_r E_c f'^3 + N_b\theta'\phi' + N_n\chi'\theta' + N_t\theta'^2\right], \quad (32)$$

$$\Phi = \phi'' + S_c f\phi' + \frac{N_t}{N_b}\theta'', \quad (33)$$

$$X = \chi'' + S_b f\chi' - P_b[\phi'\chi' + \tau_\circ\phi'' + \phi''\chi] + \frac{N_t}{N_n}\theta'' + \tilde{r}S_b\chi\left(1 - \frac{\chi}{K_n}\right), \quad (34)$$

$$\Upsilon = \xi'' + S_d\left[f\xi' - A_2\xi + A_{1\chi}\right], \quad (35)$$

$$\psi = \varphi'' + S_s\left[f\varphi' + \tilde{\lambda}_s\xi\right] + \frac{N_t}{N_s}\theta''. \quad (36)$$

Before constructing the iterative formula, we need to define the coefficients of the transformed variables.

$$a_{0r} = \frac{\partial F}{\partial f'''} = 1, a_{1r} = \frac{\partial F}{\partial f''} = f_r, a_{2r} = \frac{\partial F}{\partial f'} = -\left(M + K_p + 2F_r f_r\right), a_{3r} = \frac{\partial F}{\partial f} = f'', a_{4r} = \frac{\partial F}{\partial \theta} = \frac{G_r}{Re^2},$$

$$a_{5r} = \frac{\partial F}{\partial \phi} = -\frac{G_r N_r}{Re^2}, a_{6r} = \frac{\partial F}{\partial \chi} = -\frac{G_r R_b}{Re^2},$$

$$b_{0r} = \frac{\partial \Theta}{\partial \theta''} = 1 + \frac{4}{3}R_d, b_{1r} = \frac{\partial \Theta}{\partial \theta'} = P_r f_r + P_r N_b \phi_r' + P_r N_n \chi_r' + 2P_r N_t \theta_r', b_{2r} = \frac{\partial \Theta}{\partial f''} = 2P_r E_c f_r'',$$

$$b_{3r} = \frac{\partial \Theta}{\partial f'} = 2P_r M E_c f_r' + 2P_r K_P E_c f_r' + 3P_r F_r E_c f_r'^2, b_{4r} = \frac{\partial \Theta}{\partial f} = P_r\theta_r, b_{5r} = \frac{\partial \Theta}{\partial \phi'} = P_r N_b \theta_r',$$

$$b_{6r} = \frac{\partial \Theta}{\partial \chi'} = P_r N_n \theta_r'$$

$$c_{0r} = \frac{\partial \Phi}{\partial \phi''} = 1, c_{1r} = \frac{\partial \Phi}{\partial \phi'} = S_c f_r, \qquad c_{2r} = \frac{\partial \Phi}{\partial f} = S_c\phi'', c_{3r} = \frac{\partial \Phi}{\partial \phi''} = \frac{N_t}{N_b}$$

$$d_{0r} = \frac{\partial X}{\partial \chi''} = 1, d_{1r} = \frac{\partial X}{\partial \chi'} = S_b f_r - P_b\phi_r', d_{2r} = \frac{\partial X}{\partial \chi} = -P_b\phi_r'' + \tilde{r}S_b\left(1 - \frac{2\chi_r}{K_n}\right), d_{3r} = \frac{\partial X}{\partial f} = S_b\chi_r',$$

$$d_{4r} = \frac{\partial X}{\partial \theta''} = \frac{N_t}{N_n}, d_{5r} = \frac{\partial X}{\partial \phi''} = -P_b(\tau_o + \chi_r), d_{6r} = \frac{\partial X}{\partial \phi'} = -P_b\chi_r',$$

$$e_{0r} = \frac{\partial \Upsilon}{\partial \xi''} = 1, e_{1r} = \frac{\partial \Upsilon}{\partial \xi'} = S_d f_r,\ e_{2r} = \frac{\partial \Upsilon}{\partial \xi} = S_d A_2,\ e_{3r} = \frac{\partial \Upsilon}{\partial f} = S_d\xi_r',\ e_{4r} = \frac{\partial \Upsilon}{\partial \chi} = S_d A_1,$$

$$h_{0r} = \frac{\partial \psi}{\partial \varphi''} = 1, h_{1r} = \frac{\partial \psi}{\partial \varphi'} = S_s f_r, h_{2r} = \frac{\partial \psi}{\partial f} = S_s\varphi_r', h_{3r} = \frac{\partial \psi}{\partial \theta''} = \frac{N_t}{N_s}, h_{4r} = \frac{\partial \psi}{\partial \xi} = S_s\tilde{\lambda}_s. \quad (37)$$

We then define the residuals and also construct the iterative formula for the system of equations

$$a_{0r}f'''_{r+1} + a_{1r}f''_{r+1} + a_{2r}f'_{r+1} + a_{3r}f_{r+1} + a_{4r}\theta_{r+1} + a_{5r}\phi_{r+1} + a_{6r}\chi_{r+1} - F = R_f, \quad (38)$$

$$b_{0r}\theta''_{r+1} + b_{1r}\theta'_{r+1} + b_{2r}f''_{r+1} + b_{3r}f'_{r+1} + b_{4r}f_{r+1} + b_{5r}\phi'_{r+1} + b_{6r}\chi'_{r+1} - \Theta = R_\Theta, \quad (39)$$

$$c_{0r}\phi''_{r+1} + c_{1r}\phi'_{r+1} + c_{2r}f_{r+1} + c_{3r}\theta''_{r+1} - \Phi = R_\Phi, \quad (40)$$

$$d_{0r}\chi''_{r+1} + d_{1r}\chi'_{r+1} + d_{2r}\chi_{r+1} + d_{3r}f_{r+1} + d_{4r}\theta''_{r+1} + d_{5r}\phi''_{r+1} + d_{6r}\phi'_{r+1} - X = R_X, \quad (41)$$

$$e_{0r}\xi''_{r+1} + e_{1r}\xi'_{r+1} + e_{2r}\xi_{r+1} + e_{3r}f_{r+1} + e_{4r}\chi_{r+1} - \Upsilon = R_\Upsilon, \quad (42)$$

$$h_{0r}\varphi''_{r+1} + h_{1r}\varphi'_{r+1} + h_{2r}f_{r+1} + h_{3r}\theta''_{r+1} + h_{4r}\xi_{r+1} - \psi = R_\psi. \quad (43)$$

The iterative boundary conditions are given by

$$f'_{r+1}(0) = 1, f_{r+1}(0) = 0, f'_{r+1}(\infty) \to 0, f''_{r+1}(\infty) \to 0, \theta'_{r+1}(0) = -B_i\big(1 - \theta_{r+1}(0)\big), \theta_{r+1}(\infty) \to 0,$$

$$N_b\phi'_{r+1}(0) + N_t\theta'_{r+1}(0) = 0, \phi_{r+1}(\infty) \to 0, \chi'_{r+1}(0) = -B_n\big(1 - \chi_{r+1}(0)\big), \chi_{r+1}(\infty) \to 0, \xi_{r+1}(0) = 1,$$

$$\xi_{r+1}(\infty) \to 0, \varphi_{r+1}(0) \to 1, \varphi_{r+1}(\infty) \to 0. \quad (44)$$

Assume f_o, θ_o, ϕ_o, χ_o, ξ_o and φ_o are initial guesses such that

$$f_o(\eta) = 1 - e^{-\eta}, \theta_o(\eta) = \frac{B_i}{1+B_i}e^{-\eta},\ \phi_o(\eta) = -\frac{N_t}{N_b}\frac{B_i}{1+B_i}e^{-\eta},\ \chi_o(\eta) = \frac{B_n}{1+B_n}e^{-\eta},\ \xi_o(\eta) = e^{-\eta},$$

$$\varphi_o(\eta) = e^{-\eta}. \quad (45)$$

Applying equation 31,

$$(a_{0r}D^3 + a_{1r}D^2 + a_{2r}D + a_{3r}I)f_{r+1} + (a_{4r}I)\theta_{r+1} + (a_{5r}I)\phi_{r+1} + (a_{6r}I)\chi_{r+1} + 0\xi_{r+1} + 0\varphi_{r+1} = R_f \quad (46)$$

$$(b_{2r}D^2 + b_{3r}D + b_{4r}I)f_{r+1} + (b_{0r}D^2 + b_{1r}D)\theta_{r+1} + (b_{5r}D)\phi_{r+1} + (a_{6r}D)\chi_{r+1} + 0\xi_{r+1} + 0\varphi_{r+1} = R_\Theta \quad (47)$$

$$(c_{2r}I)f_{r+1} + (c_{3r}D^2)\theta_{r+1} + (c_{0r}D^2 + c_{1r}D)\phi_{r+1} + 0\chi_{r+1} + 0\xi_{r+1} + 0\varphi_{r+1} = R_\Phi \quad (48)$$

$$(d_{3r}I)f_{r+1} + (d_{4r}D^2)\theta_{r+1} + (d_{5r}D^2 + d_{6r}D)\phi_{r+1} + (d_{0r}D^2 + d_{1r}D + d_{2r}I)\chi_{r+1} + 0\xi_{r+1} + 0\varphi_{r+1} = R_X \quad (49)$$

$$(e_{3r}D)f_{r+1} + 0\theta_{r+1} + 0\phi_{r+1} + (e_{4r}I)\chi_{r+1} + (e_{0r}D^2 + e_{1r}D + e_{2r}I)\xi_{r+1} + 0\varphi_{r+1} = R_\xi \quad (50)$$

$$(h_{2r}I)f_{r+1} + (d_{3r}D^2)\theta_{r+1} + 0\phi_{r+1} + 0\chi_{r+1} + (h_{4r}I)\xi_{r+1} + (h_{0r}D^2 + h_{1r}D)\varphi_{r+1} = R_\psi \quad (51)$$

Equations 46 - 51 can be written in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} \begin{bmatrix} f_{r+1} \\ \theta_{r+1} \\ \phi_{r+1} \\ \chi_{r+1} \\ \xi_{r+1} \\ \varphi_{r+1} \end{bmatrix} = \begin{bmatrix} R_f \\ R_\Theta \\ R_\Phi \\ R_X \\ R_\Upsilon \\ R_\psi \end{bmatrix} \tag{53}$$

where

$A_{11} = a_{0r}D^3 + a_{1r}D^2 + a_{2r}D + a_{3r}I, A_{12} = a_{4r}I, A_{13} = a_{5r}I, A_{14} = a_{6r}I, A_{15} = 0, A_{16} = 0, A_{21} = b_{2r}D^2 + b_{3r}D + b_{4r}I, A_{22} = b_{0r}D^2 + b_{1r}D, A_{23} = b_{5r}D, A_{24} = a_{6r}D, A_{25} = 0, A_{26} = 0, A_{31} = c_{2r}I, A_{32} = c_{3r}D^2, A_{33} = c_{0r}D^2 + c_{1r}D, A_{34} = 0, A_{35} = 0, A_{36} = 0, A_{41} = d_{3r}I, A_{42} = d_{4r}D^2, A_{43} = d_{5r}D^2 + d_{6r}D, A_{44} = d_{0r}D^2 + d_{1r}D + d_{2r}I, A_{45} = 0, A_{46} = 0, A_{51} = e_{3r}D, A_{52} = 0, A_{53} = 0, A_{54} = e_{4r}I, A_{55} = e_{0r}D^2 + e_{1r}D + e_{2r}I, A_{56} = 0, A_{61} = h_{2r}I, A_{62} = d_{3r}D^2, A_{63} = 0, A_{64} = 0, A_{65} = h_{4r}I, A_{66} = h_{0r}D^2 + h_{1r}D.$

Results and Discussion

In this section, we discuss some key findings from the current study. Table 1 gives the values of the parameters used for the simulations and their sources. To validate the accuracy of the numerical scheme, we perform a residual error analysis. The results for this are given in Table 2 and Figure 1. Our errors are in the orders of $10^{-12} - 10^{-10}$, which means our scheme, is very accurate.

Table 1: List of dimensionless parameters

Parameter	Symbol	Value	Source
Magnetic field	M	(0, 6)	24, 25
Permeability	K_p	0.2	26
Forchheimer number	F_r	0.5	27
Grashof number	G_r	(0.1, 1)	26
Reynolds number	Re	1	26
Buoyancy ratio	N_r	(0.3, 1.2)	28
Bioconvective Rayleign number	R_b	0.1	8
Thermal radiation	R_d	0.3	9
Prandth number	P_r	0.72	27
Eckert number	E_c	(0.1, 3)	29
Brownian motion	N_b	(0.1, 5)	30
Thermophoresis	N_t	(0.1, 0.5)	30, 6
Microbe's Brownian motion	N_n	0.5	31
Schimdst number	S_c	3.4	31
Bioconvective Schimdst number	S_d	3.4	Assumed
Bioconvective Peclet number	P_b	(0.1, 1.0)	32, 9
Microbial difference constant	τ_o	0.2	8
Microbial growth rate	r	0.1	33
Carrying capacity	K_n	10	Assumed
Schmidt number for the protein	S_d	3.4	Assumed

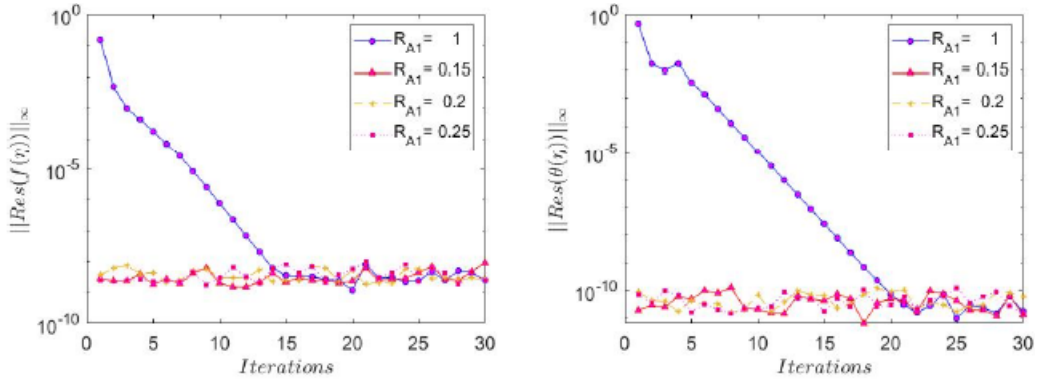
Schmidt number for the product	S_s	3.4	Assumed
Exoprotein production rate	A_1	0.2	34
Natural degradation of exoprotein	A_2	0.1	Assumed
Production rate for the product	λ_s	0.9	Assumed
Biot number (Thermal)	B_i	(0.1, 50)	35, 24
Biot number (Microbial)	B_n	(0.1, 50)	Assumed

Table 2: Residual Errors

i	$f(\eta)$	$\theta(\eta)$	$\phi(\eta)$	$\chi(\eta)$	$\xi(\eta)$	$\varphi(\eta)$
1	$4.52879190 \times 10^{-09}$	$2.48885357 \times 10^{-11}$	$2.98805425 \times 10^{-12}$	$1.89686045 \times 10^{-11}$	$4.66642003 \times 10^{-12}$	$1.75433622 \times 10^{-11}$
2	$2.30051267 \times 10^{-09}$	$5.62727642 \times 10^{-12}$	$3.44213547 \times 10^{-12}$	$6.90081325 \times 10^{-12}$	$5.01332309 \times 10^{-12}$	$1.15610854 \times 10^{-11}$
3	$3.69907971 \times 10^{-09}$	$1.67381664 \times 10^{-11}$	$4.39359660 \times 10^{-12}$	$1.16373577 \times 10^{-11}$	$5.10451403 \times 10^{-12}$	$2.86101143 \times 10^{-11}$
4	$1.31282119 \times 10^{-09}$	$1.42612588 \times 10^{-11}$	$3.60267371 \times 10^{-12}$	$1.06941123 \times 10^{-11}$	$5.74022774 \times 10^{-12}$	$1.46910817 \times 10^{-11}$
5	$1.89027927 \times 10^{-09}$	$1.24300570 \times 10^{-11}$	$4.64073224 \times 10^{-12}$	$1.65438774 \times 10^{-11}$	$6.08341155 \times 10^{-12}$	$1.27341471 \times 10^{-11}$
6	$3.15445683 \times 10^{-10}$	$2.46913601 \times 10^{-11}$	$4.51194637 \times 10^{-12}$	$1.28380639 \times 10^{-11}$	$5.21832577 \times 10^{-12}$	$1.54740110 \times 10^{-11}$
7	$4.22095026 \times 10^{-09}$	$2.02327044 \times 10^{-11}$	$3.43169937 \times 10^{-12}$	$7.41717798 \times 10^{-12}$	$4.05986356 \times 10^{-12}$	$2.35455266 \times 10^{-11}$
8	$6.56893429 \times 10^{-10}$	$1.12234666 \times 10^{-11}$	$1.33004718 \times 10^{-12}$	$6.43274323 \times 10^{-12}$	$6.89298618 \times 10^{-12}$	$1.53696778 \times 10^{-11}$
9	$7.16446458 \times 10^{-10}$	$1.18538512 \times 10^{-11}$	$5.67790259 \times 10^{-12}$	$6.94899693 \times 10^{-12}$	$4.72447081 \times 10^{-12}$	$1.26509359 \times 10^{-11}$
10	$8.59152749 \times 10^{-10}$	$1.48197010 \times 10^{-11}$	$3.07043280 \times 10^{-12}$	$1.49379398 \times 10^{-11}$	$2.51314941 \times 10^{-12}$	$1.60303160 \times 10^{-11}$

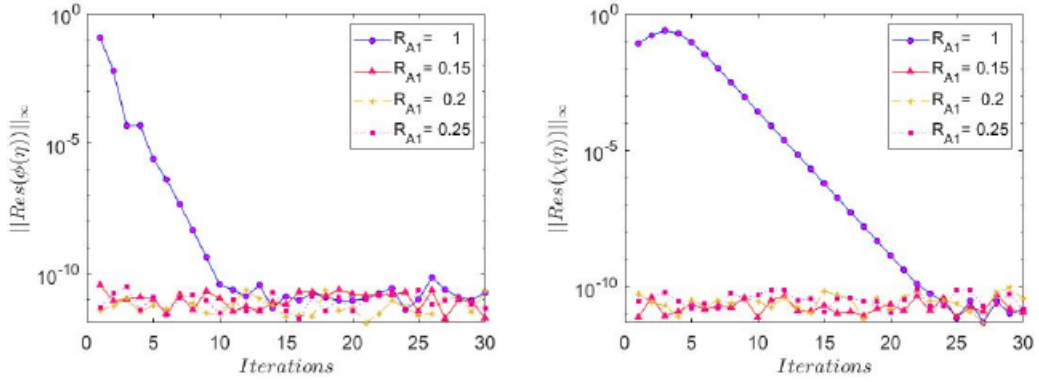
Table 3: Variation of the Skin friction, Nusselt, Sherwood, microbes density, exoprotein density, product density numbers

N_t	r	A_1	A_2	$f''(0)$	$\theta'(0)$	$\phi'(0)$	$\chi'(0)$	$\xi'(0)$	$\varphi'(0)$
0.2	0.2	0.2	0.1	−2.28480140	0.08472393	−0.12711554	−0.11929443	−0.57749971	1.95304268
0.3				−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.39091858	2.54242381
0.4				−2.28227248	0.09512741	−0.26553031	−0.04548963	−0.21920718	3.10752997
0.5				−2.28072172	0.09986325	−0.33844447	−0.01947548	−0.06190059	3.64600057
0.3	0.05			−2.27936819	0.07884141	−0.18563390	−0.31857071	−1.01332285	1.01443160
	0.1			−2.27999671	0.08010380	−0.18665849	−0.28907304	−0.93121467	1.23931874
	0.2			−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.39091858	2.54242381
	0.3			−2.29748898	0.13657905	−0.23488280	0.83715733	1.98578069	8.05432909
	0.2	0.1		−2.28363583	0.09009349	−0.19502899	−0.07848364	−0.79012017	1.48245432
		0.15		−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.59051948	2.01243891
		0.2		−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.39091858	2.54242381
		0.25		−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.19131768	3.07240871
		0.2	0.1	−2.28363583	0.09009349	−0.19502899	−0.07848362	−0.39091813	2.54242449
			0.15	−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.54240660	2.23927221
			0.2	−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.67328678	1.99496115
			0.25	−2.28363583	0.09009348	−0.19502897	−0.07848386	−0.78890232	1.79342183



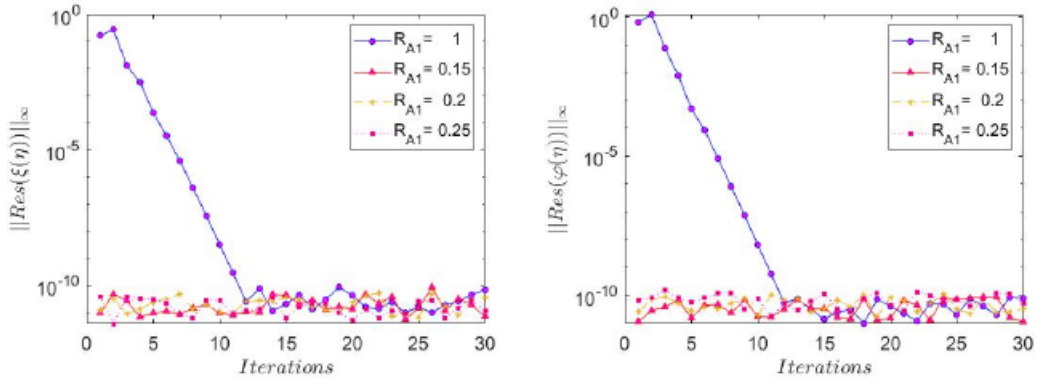
(a) Residual errors for velocity

(b) Residual errors for temperature



(c) Residual errors for nanoparticles

(d) Residual errors for microbes



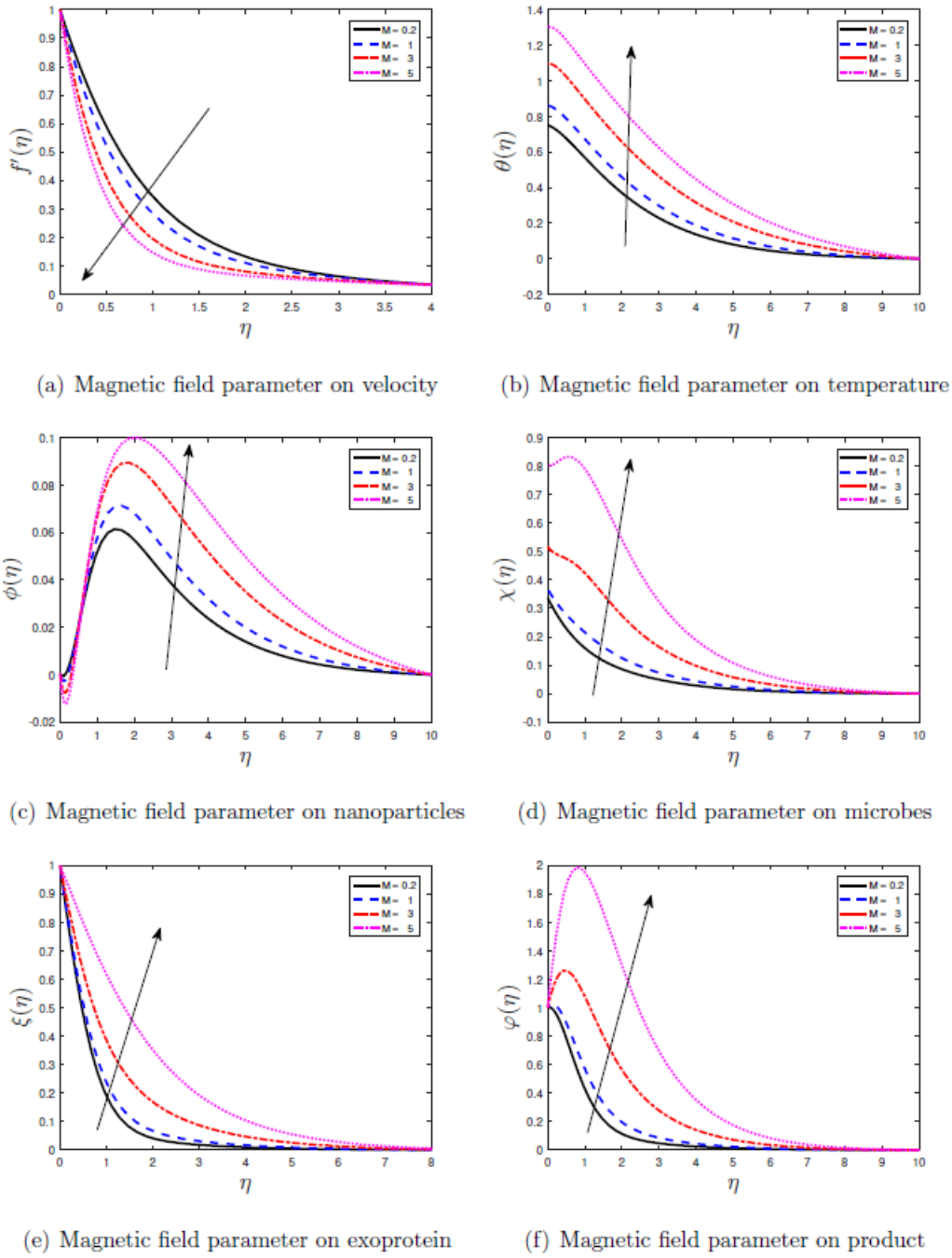
(e) Residual errors for exoprotein

(f) Residual errors for product

Source: our simulation results

Figure 1: Residual errors

In Figure (2) we study the effect of magnetic field parameters on different variables in the model. The application of a magnetic field produces a drag force that leads to a reduction in the velocity of the fluid. This is accompanied by an increase in all other species within the boundary layer. Similar results are found in the literature[36-38]. The exoprotein and product are also shown to increase at the boundary layer for increasing values of the magnetic field.



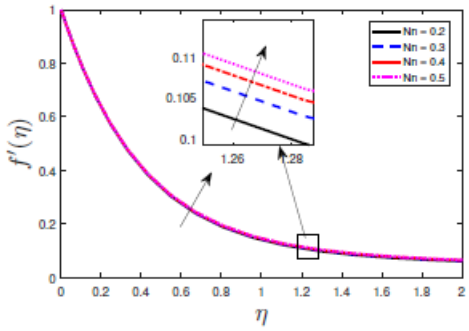
(a) Magnetic field parameter on velocity (b) Magnetic field parameter on temperature

(c) Magnetic field parameter on nanoparticles (d) Magnetic field parameter on microbes

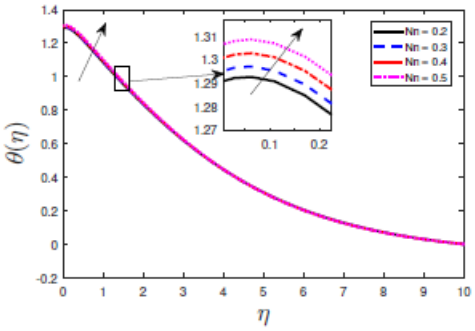
(e) Magnetic field parameter on exoprotein (f) Magnetic field parameter on product

Figure 2: Effect of magnetic field parameter

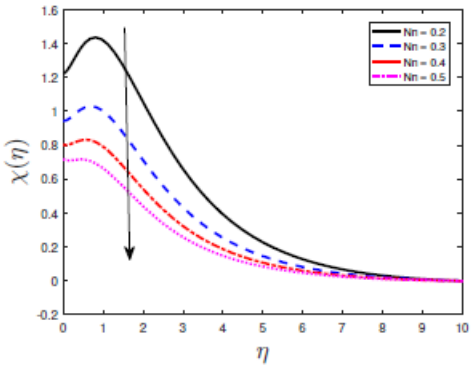
In Figure (3) we investigate the effect of the microbial Brownian motion parameter on different model variables. The microbe concentration decreases with increasing values of this parameter. Similar results for this phenomenon were presented by Dhlamini *et al*[6]. The exoprotein and product also decrease with an increase in the microbes' Brownian motion parameter. This is because these are directly dependent on the microbes, resulting in a decrease in the microbes. This, in turn, leads to a decrease in exoproteins and products. Temperature and fluid velocity increase with an increase in the microbes' Brownian motion parameter. Dhlamini *et al*[6] obtained similar results for the temperature profile.



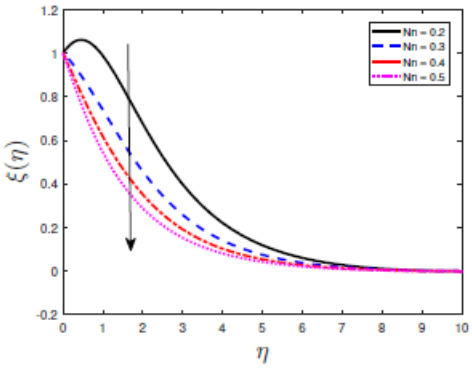
(a) Magnetic field parameter on velocity



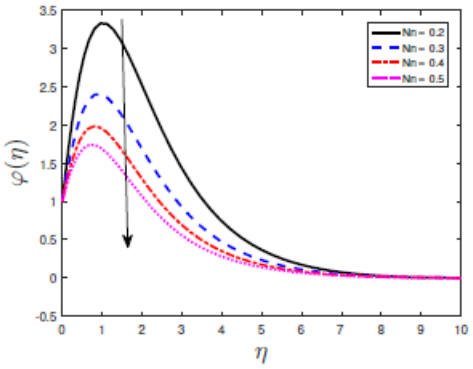
(b) Magnetic field parameter on temperature



(c) Magnetic field parameter on microbes



(d) Magnetic field parameter on exoprotein



(e) Magnetic field parameter on product

Figure 3: Effect of Microbe's Brownian motion parameter

In Figures (4), (5), and (6), we study the effect of the microbial growth rate parameter, carrying capacity, rate of production of the product, and exoprotein degradation rate. Increasing the bacterial growth rate and carrying capacity leads to an increase in the microbe population[33], this in turn increases the exoprotein and the product. The temperature also increases, since increasing the microbes' population increases the microbial Brownian motion. The solute concentration is shown to decrease with an increase in these parameters. The mechanism that drives the phenomenon is not clear at this point. An increase in exoprotein production rate is noted to have a significant effect on the product profile. The product increases significantly with an increase in the exoprotein rate[13]. Protein strains enhance the quality of products. For example, in cheese production, proteins play an

important role in determining the texture and quality of the cheese by promoting the proteolysis process and the production of bioactive peptides[39].

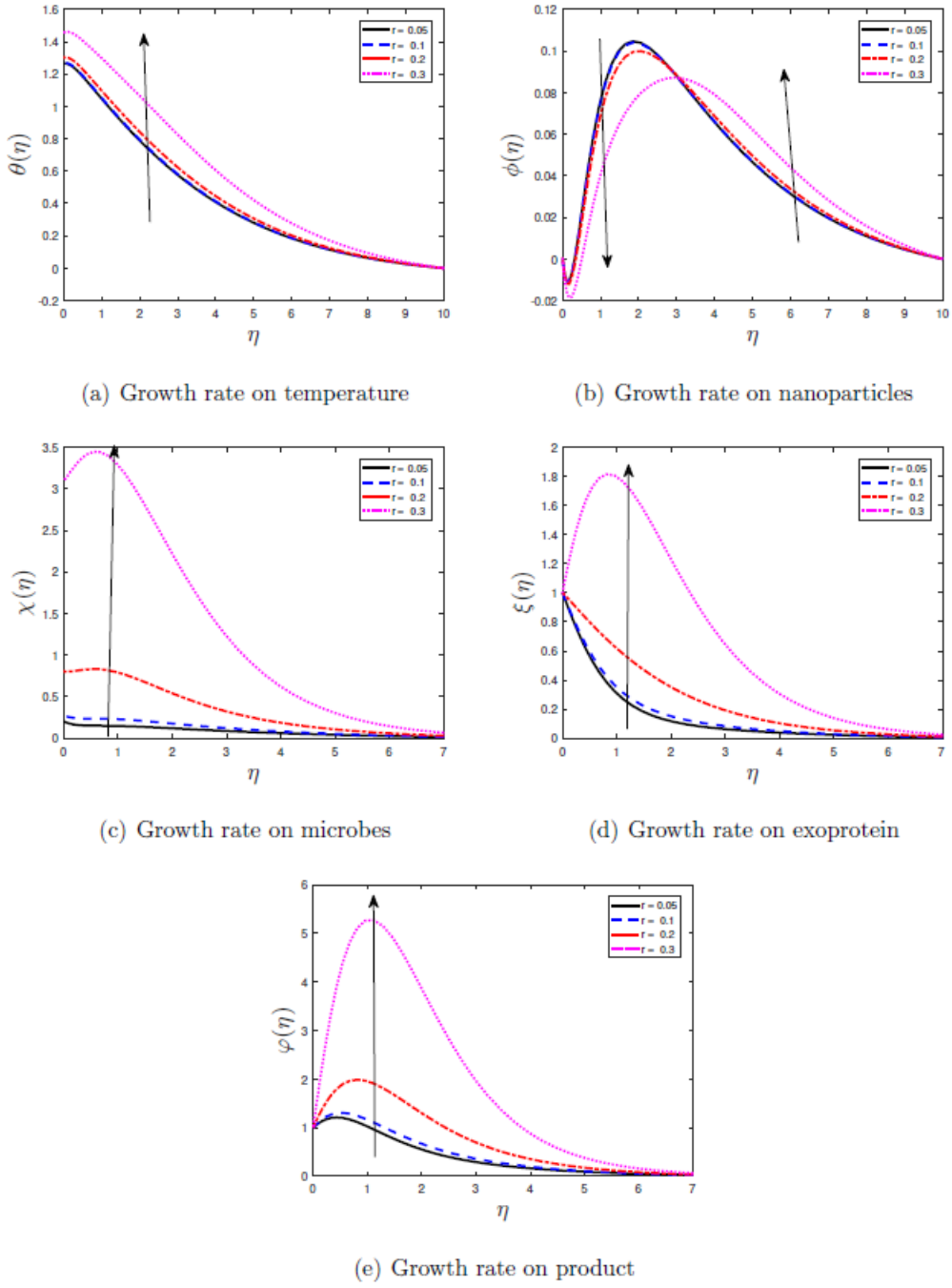
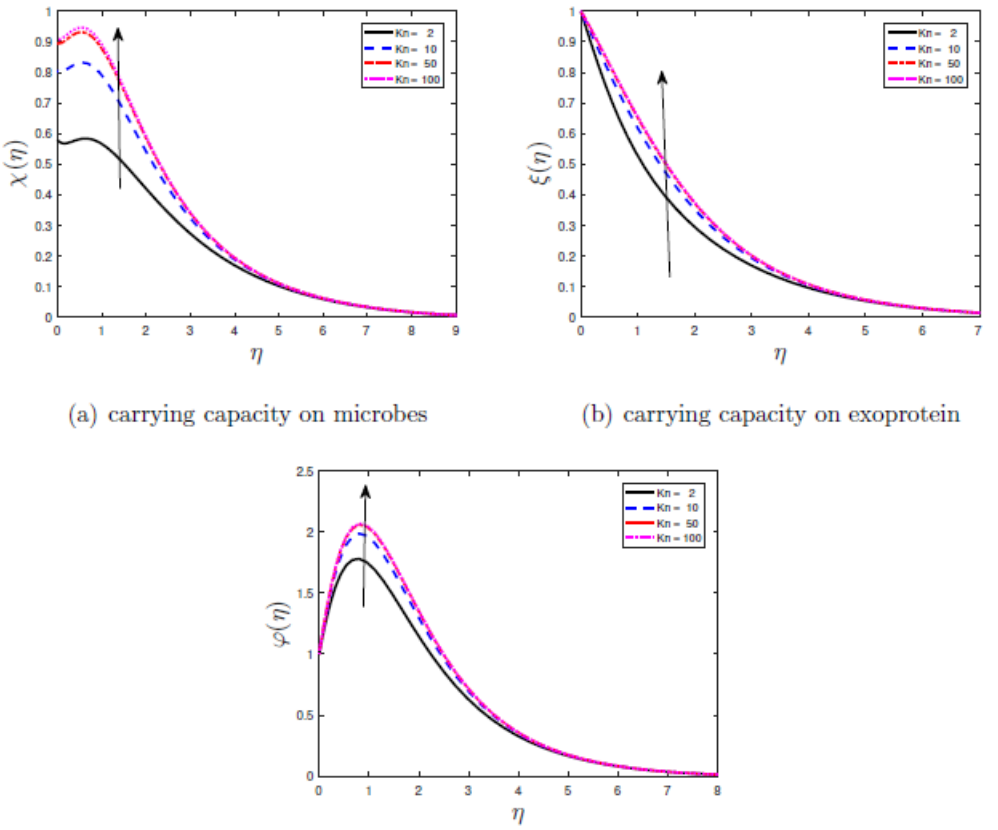


Figure 4: Effect of growth rate parameter

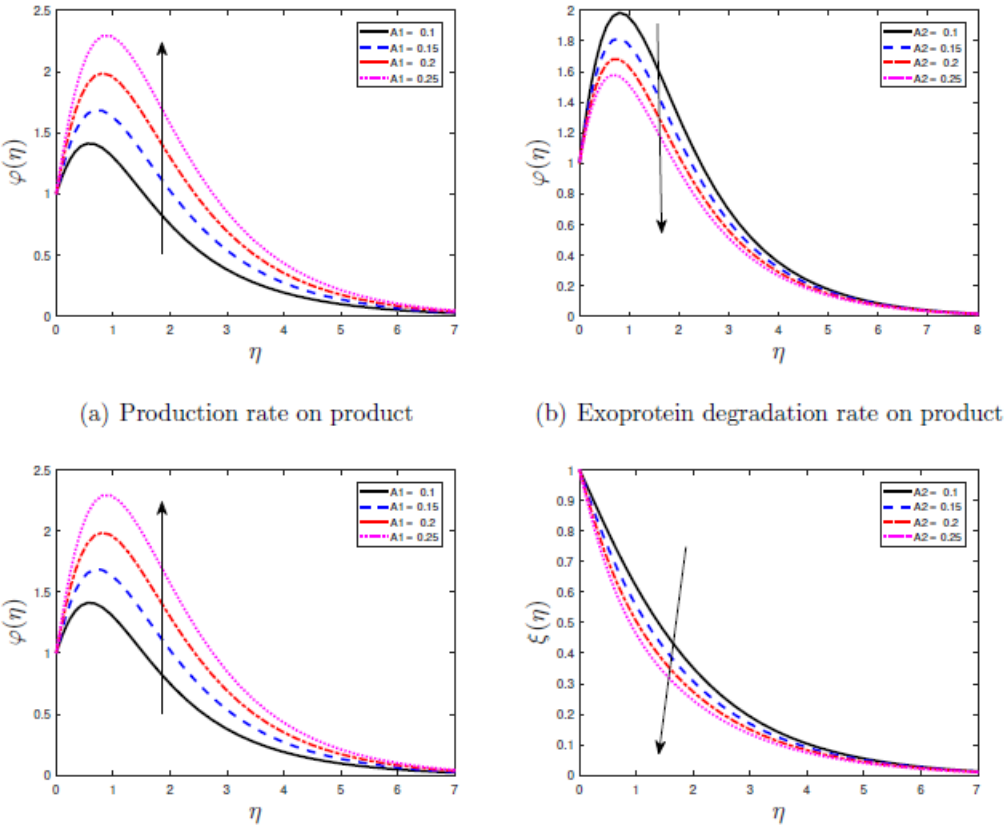


(a) carrying capacity on microbes

(b) carrying capacity on exoprotein

(c) carrying capacity on product

Figure 5: Effect of carrying capacity parameter



(a) Production rate on product

(b) Exoprotein degradation rate on product

(c) Production rate on product

(d) degradation on product

Figure 6: Effect of production and degradation rate parameters

Conclusion

In this manuscript, we formulate a model to capture the dynamics of momentum, heat, and mass transfer in the porous medium flow of a nanofluid with the effect of microbial activity. The current study extended the microbial activities to include the microbial growth and secretion of exoproteins that are used in different industrial technologies, such as in the pharmaceutical industry. The following important observations were deduced from the study:

- The magnetic field parameter retards the fluid velocity; this leads to a decrease in the convective transport and thus an increase in all other species is observed.
- Although the mechanism is not known, the growth rate parameter increases the temperature while reducing the nanoparticle concentration close to the wall.
- The microbe's Brownian motion parameter increases temperature and fluid velocity while reducing the microbes, exoproteins, and product concentration.

Data availability

No data was used in this manuscript. Simulations are based on a numerical scheme and parameter values obtained publicly in the literature. All data generated or analyzed during this study are included in this published article.

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